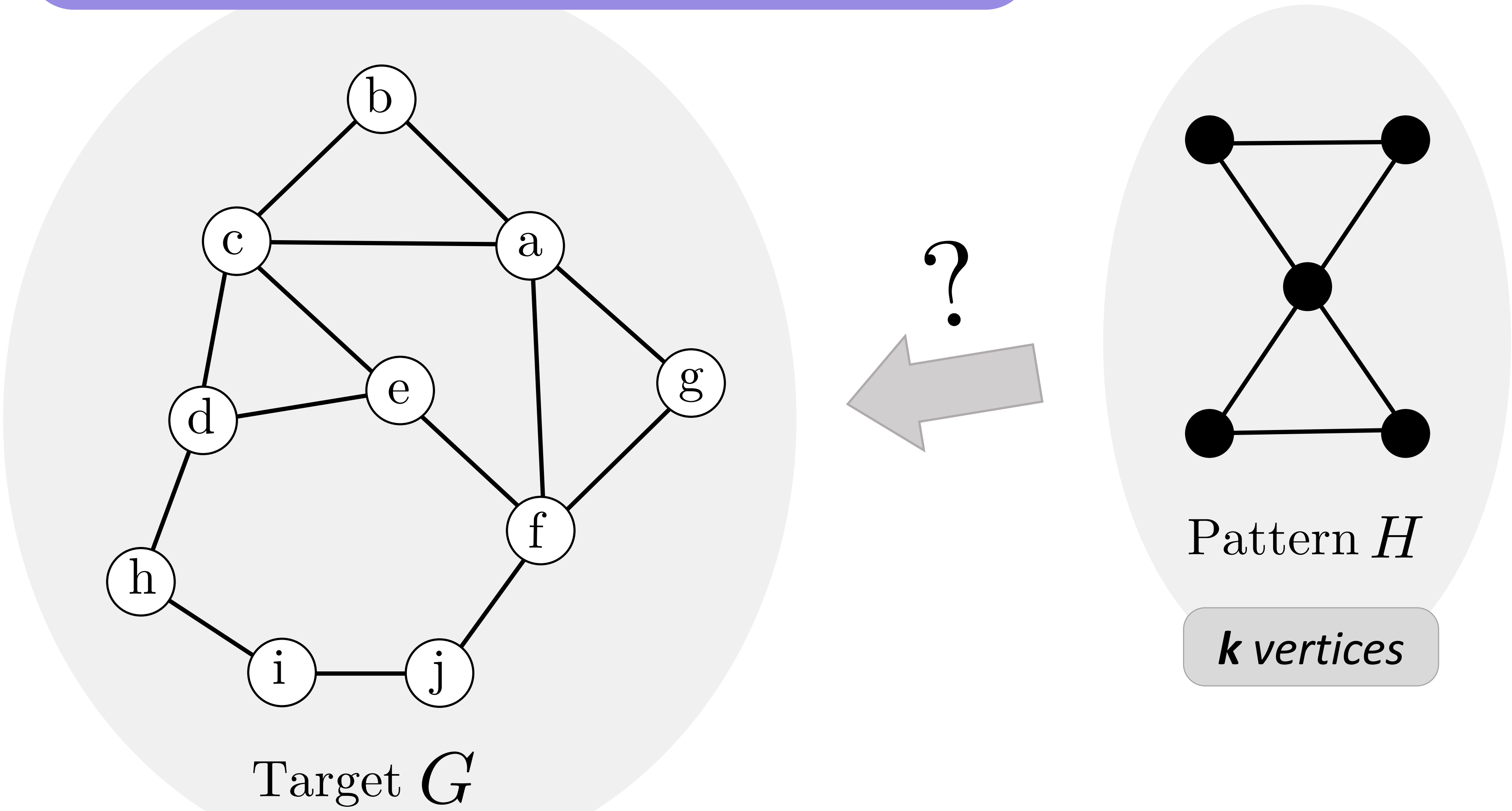
LUKAS GIANINAZZI AND TORSTEN HOEFLER

Parallel Planar Subgraph Isomorphism and Vertex Connectivity

Subgraph Isomorphism:
*Find subgraphs in the target that **match the pattern***



***n** vertices*

***k** vertices*

Subgraph Isomorphism:
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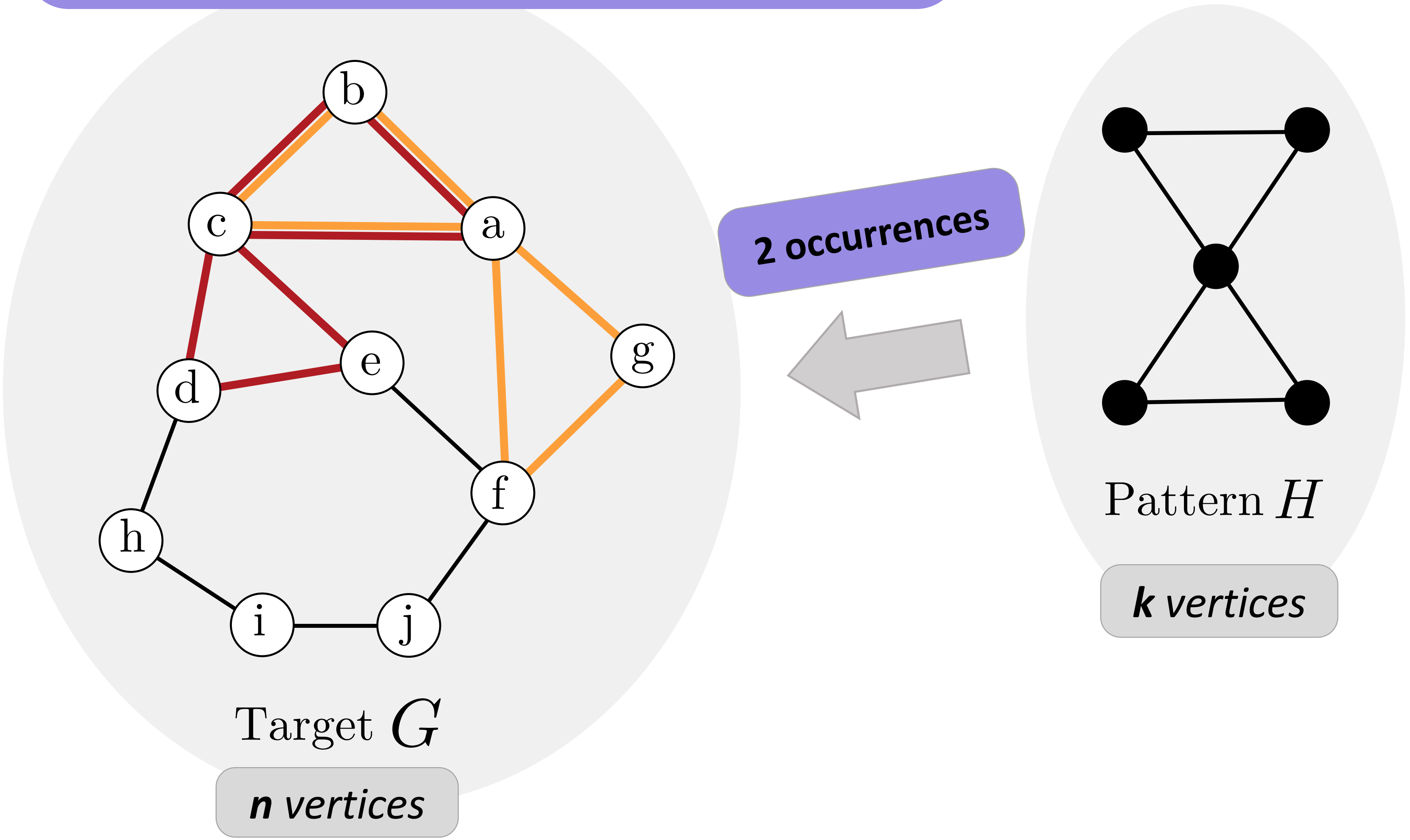

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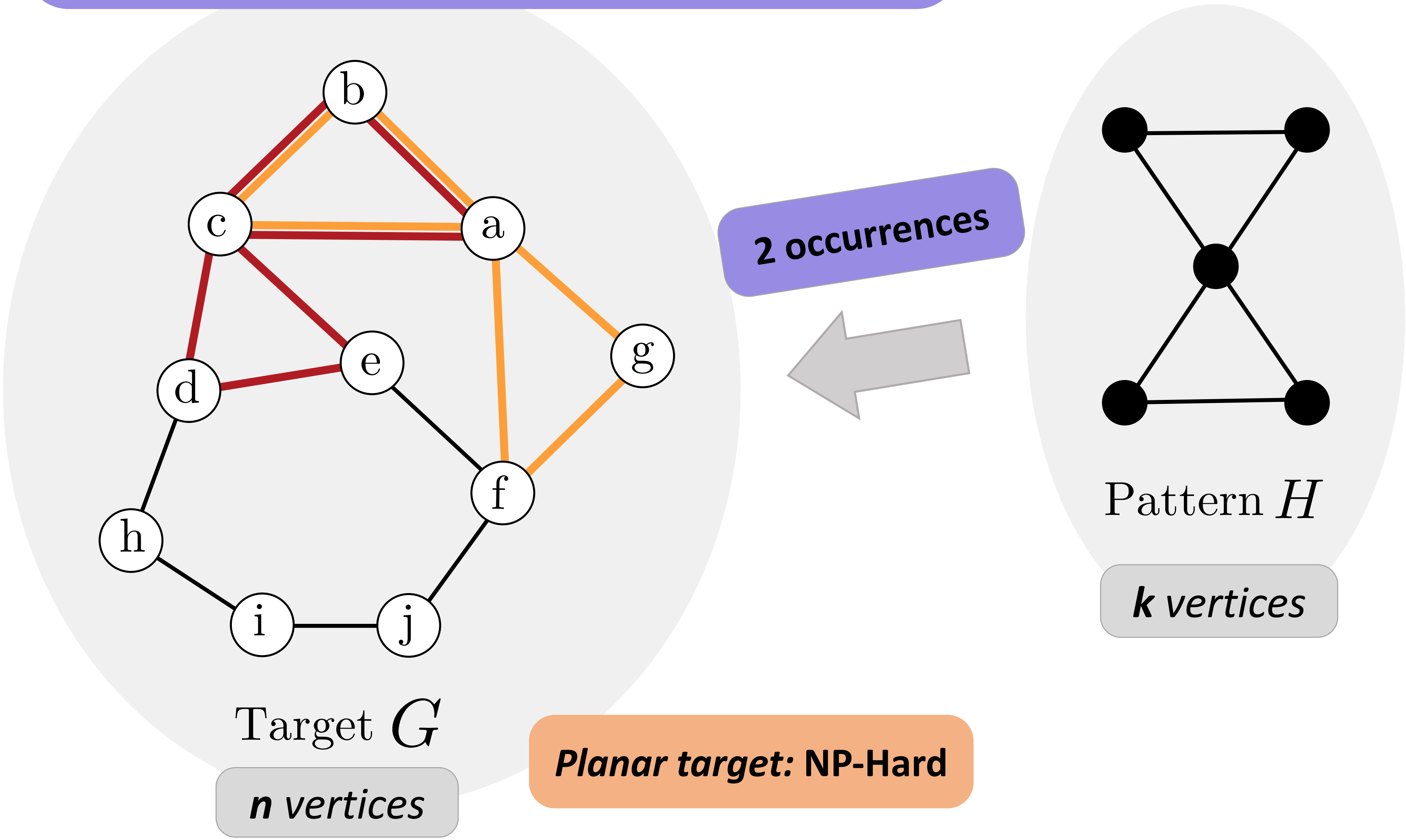

Planar target: **NP-Hard**

Subgraph Isomorphism:
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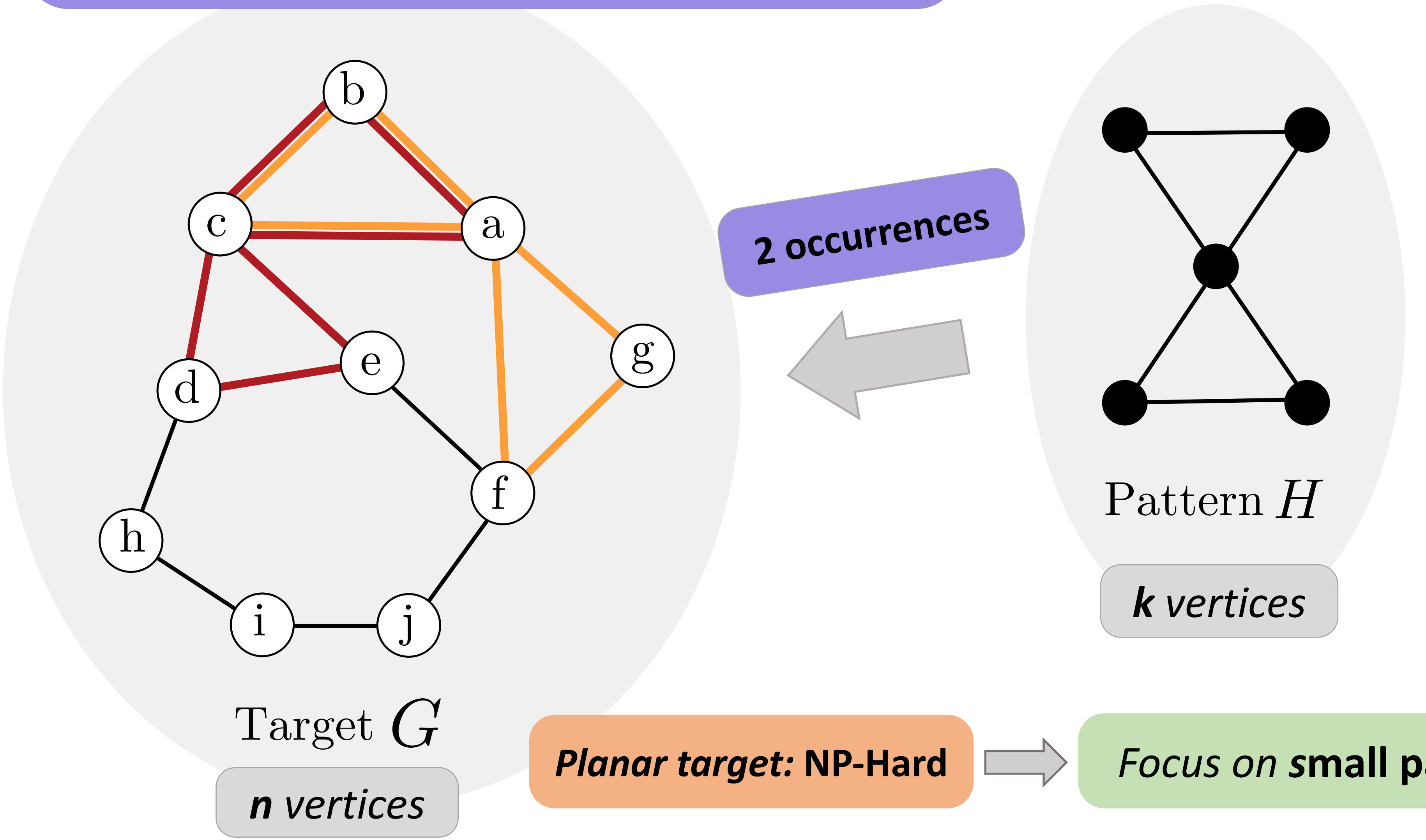
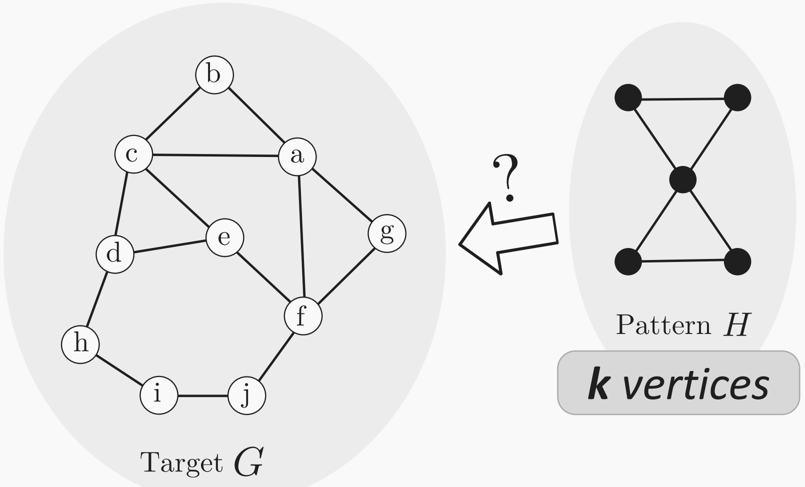


Planar target: **NP-Hard** → *Focus on* **small patterns**

Subgraph Ismorphism



n *vertices*

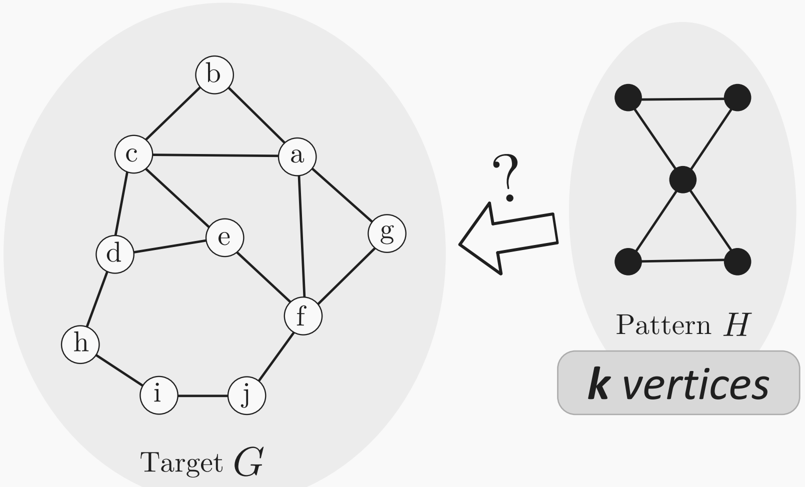
k *vertices*

Results for Planar Graphs

	Work	**Depth**
Color Coding ✱ Alon et al. 1995	$\Omega\left(n^{\sqrt{k}}\right)$	$O(\log^2 n)$

✱ *Result correct with high probability*

Subgraph Ismorphism



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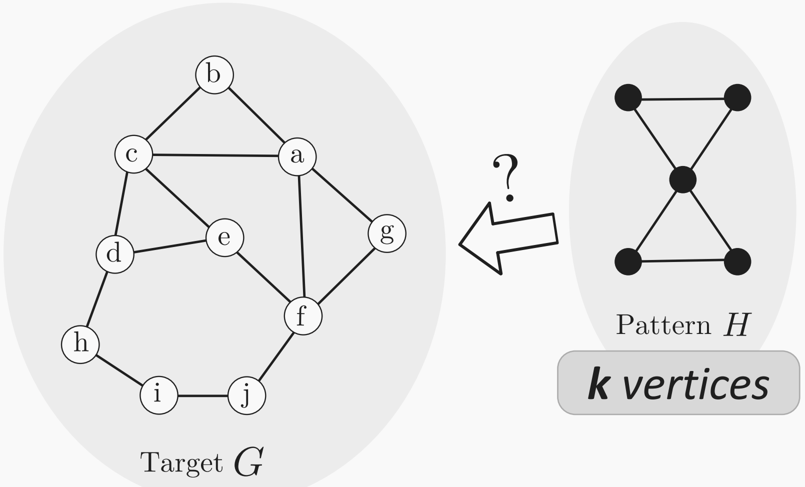
Eppstein *1995*	$O\left(k^{3k+1} n\right)$	$\Omega(n)$

* *Result correct with high probability*

Subgraph Ismorphism



***n* vertices**

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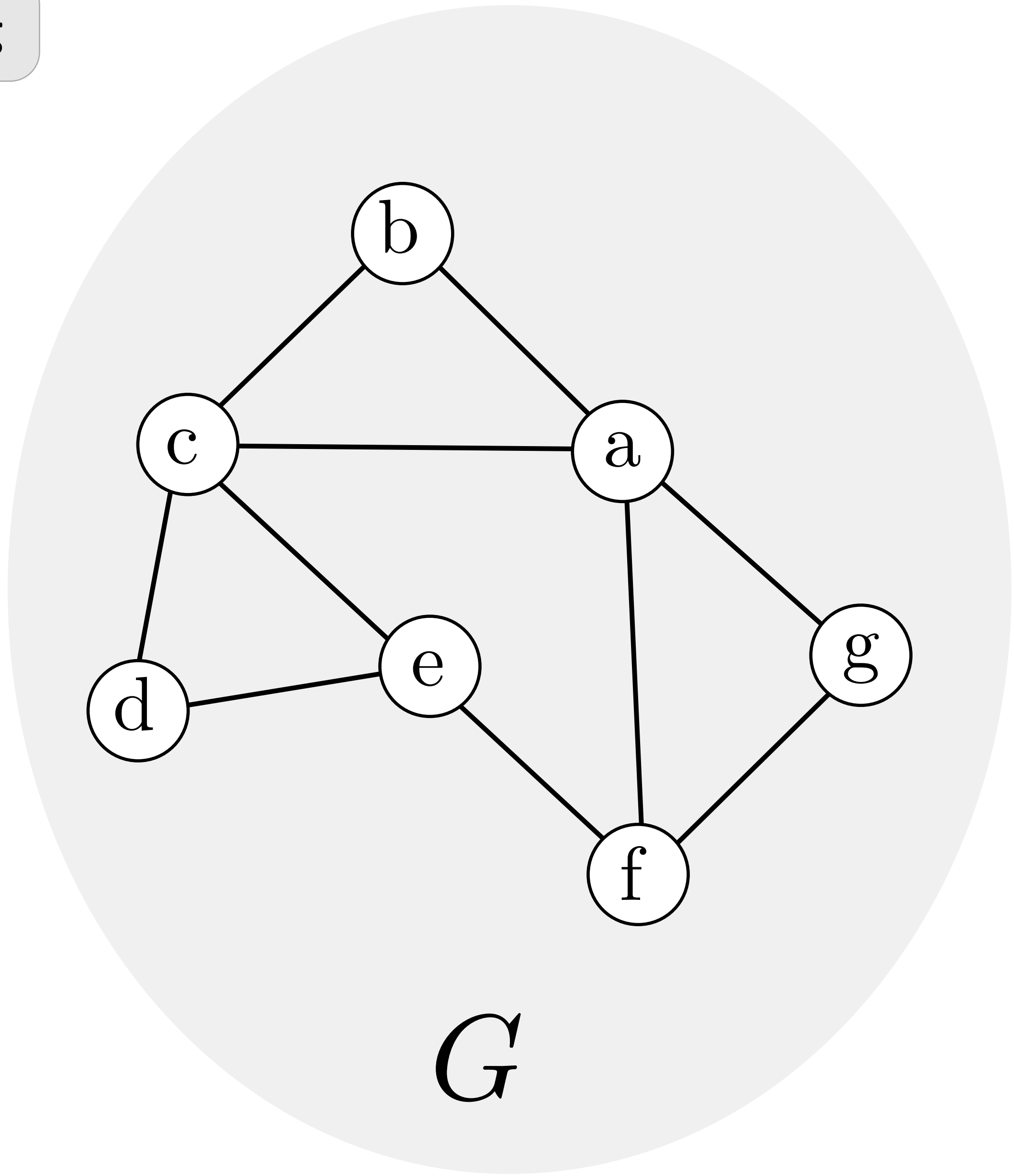
Eppstein *1995*	$O(k^{3k+1} n)$	$\Omega(n)$
Our Result ✸	$O(k^{3k+1} n \log n)$	$O(k \log n)$

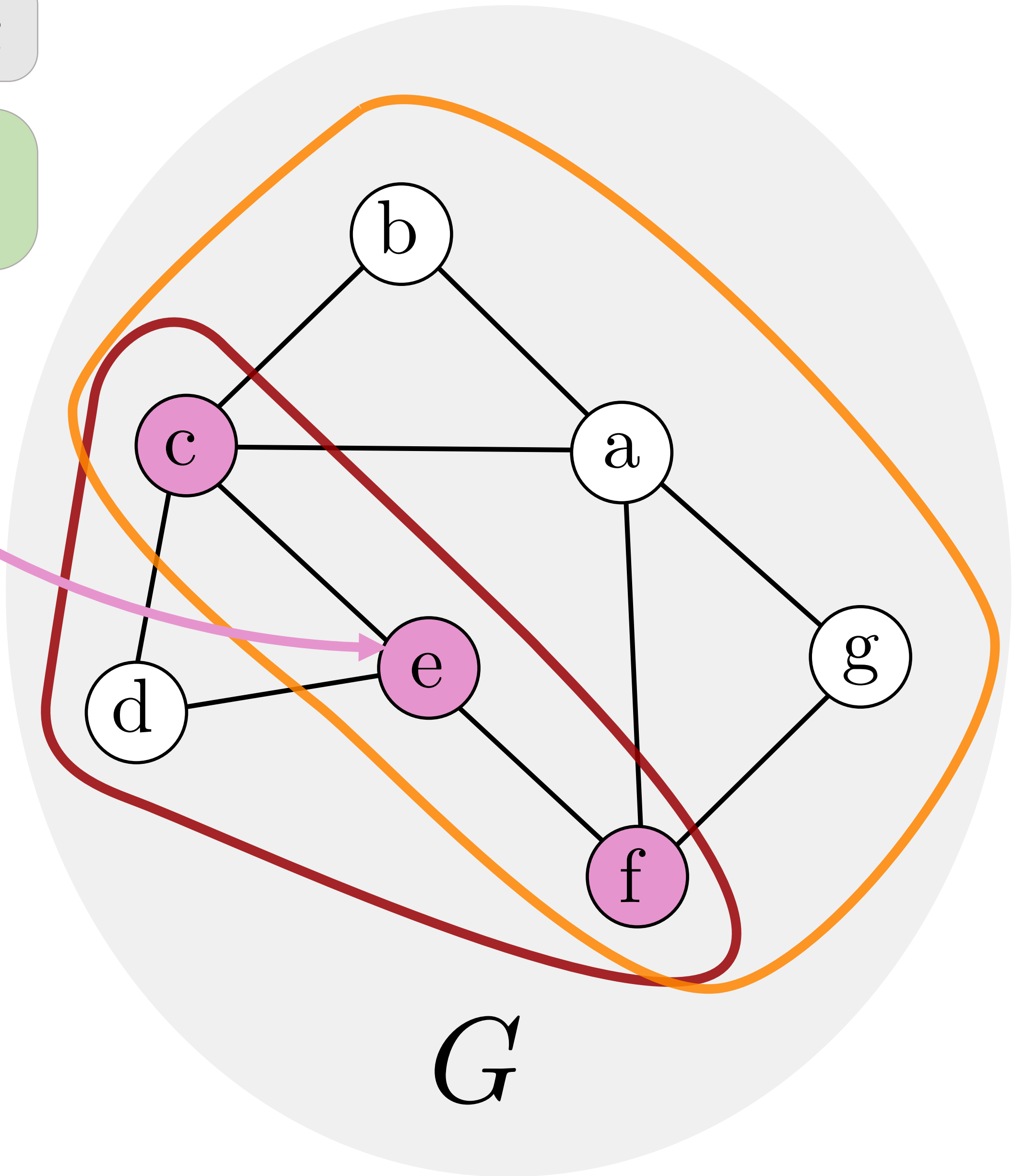
✸ *Result correct with high probability*

Dynamic Programming



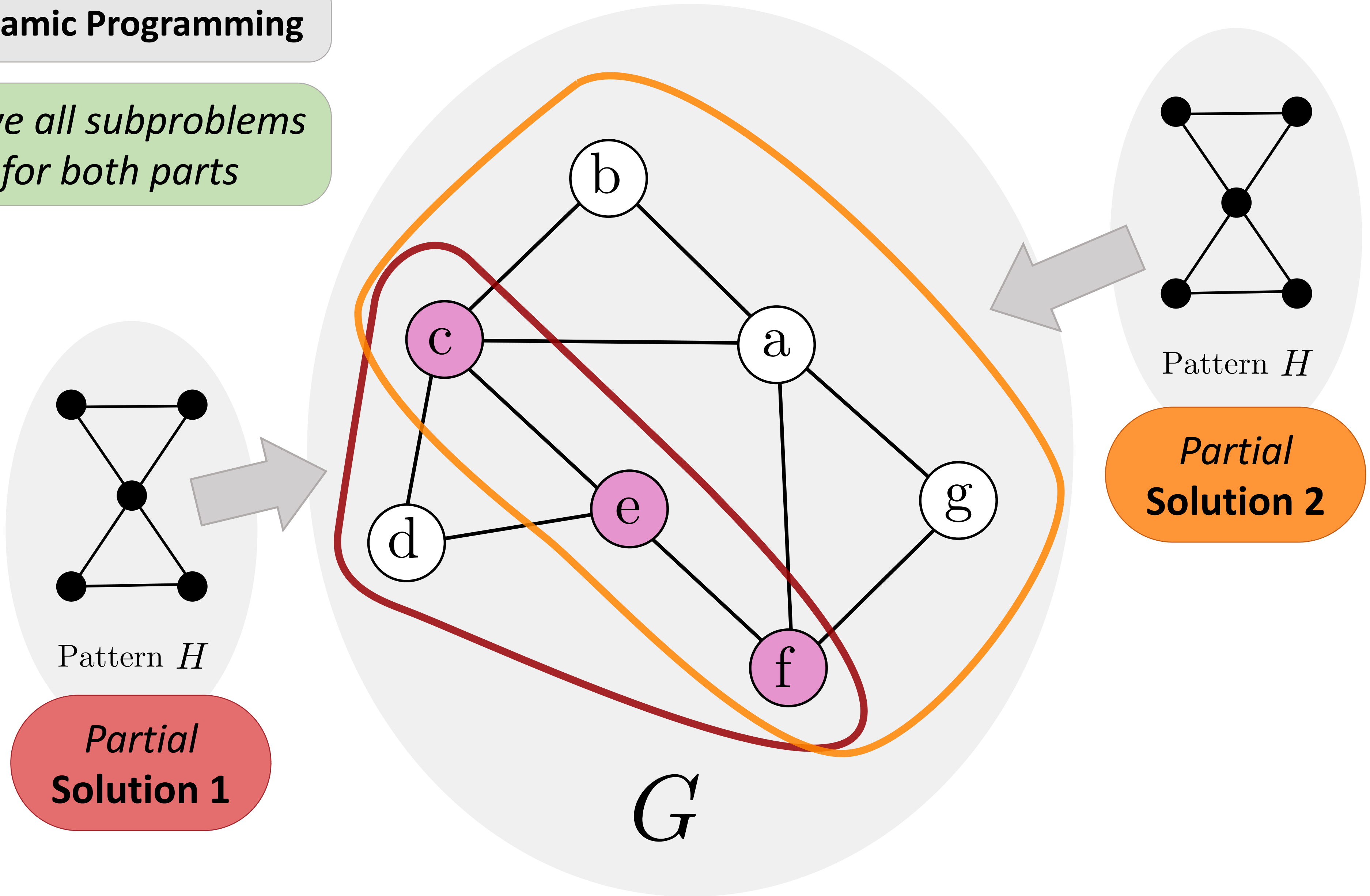
Dynamic Programming

"Shared Vertices" divide the graph



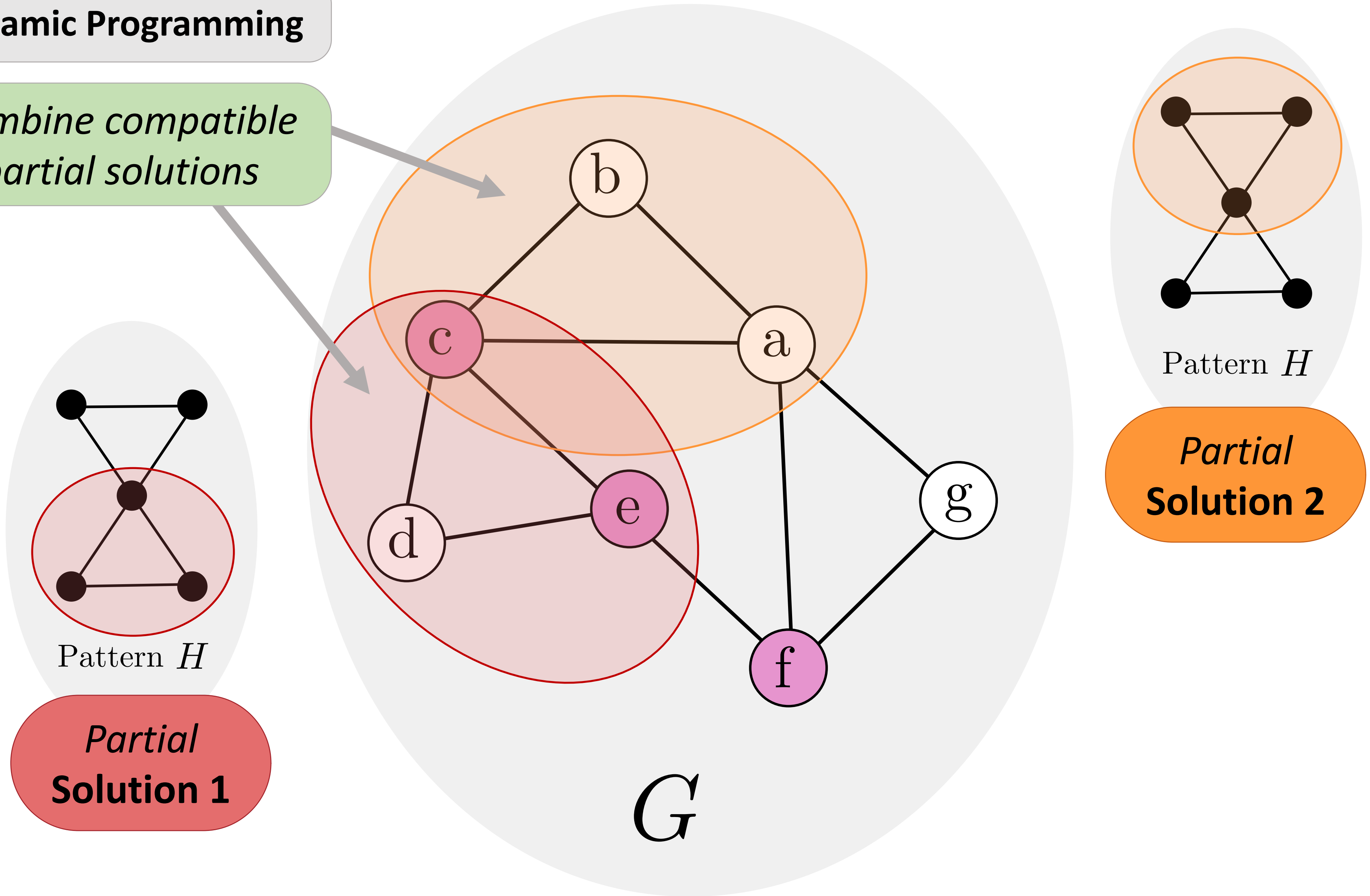
Dynamic Programming

Solve all subproblems for both parts



Dynamic Programming

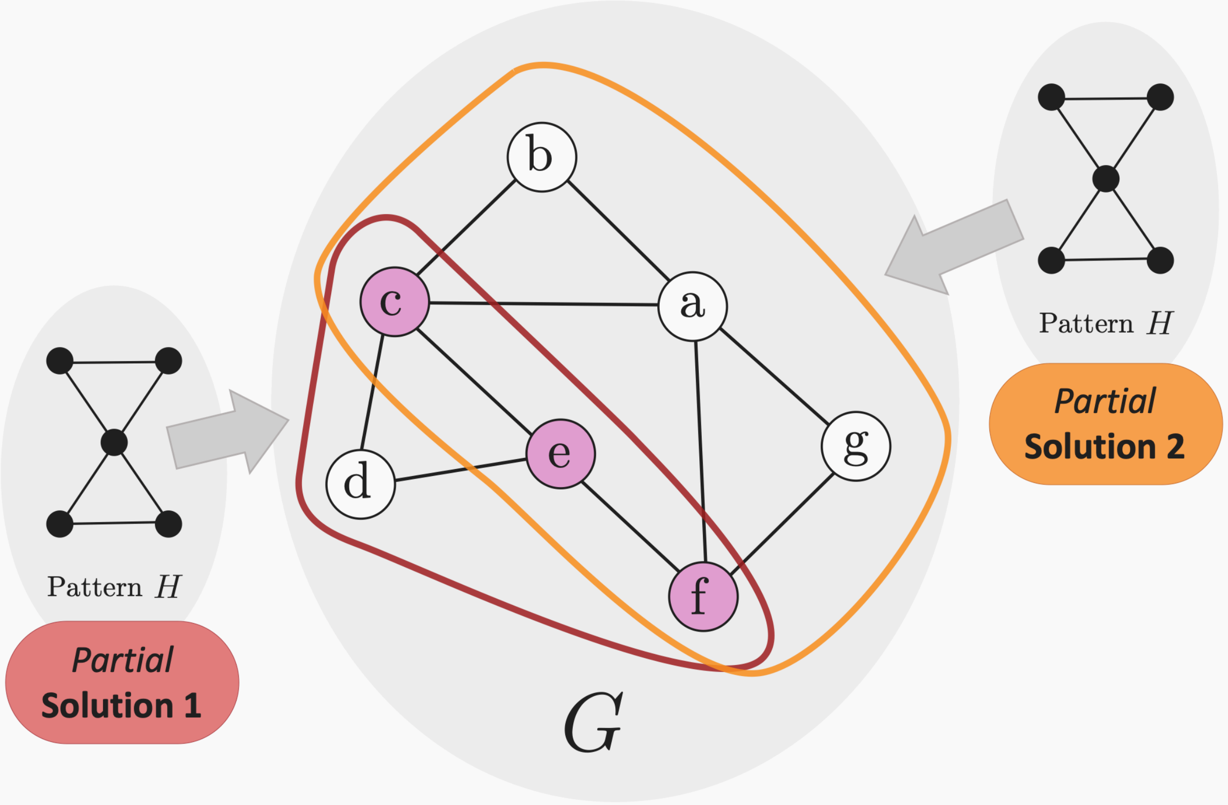
Combine compatible partial solutions



Partial **Solution 1**

Partial **Solution 2**

Dynamic Programming



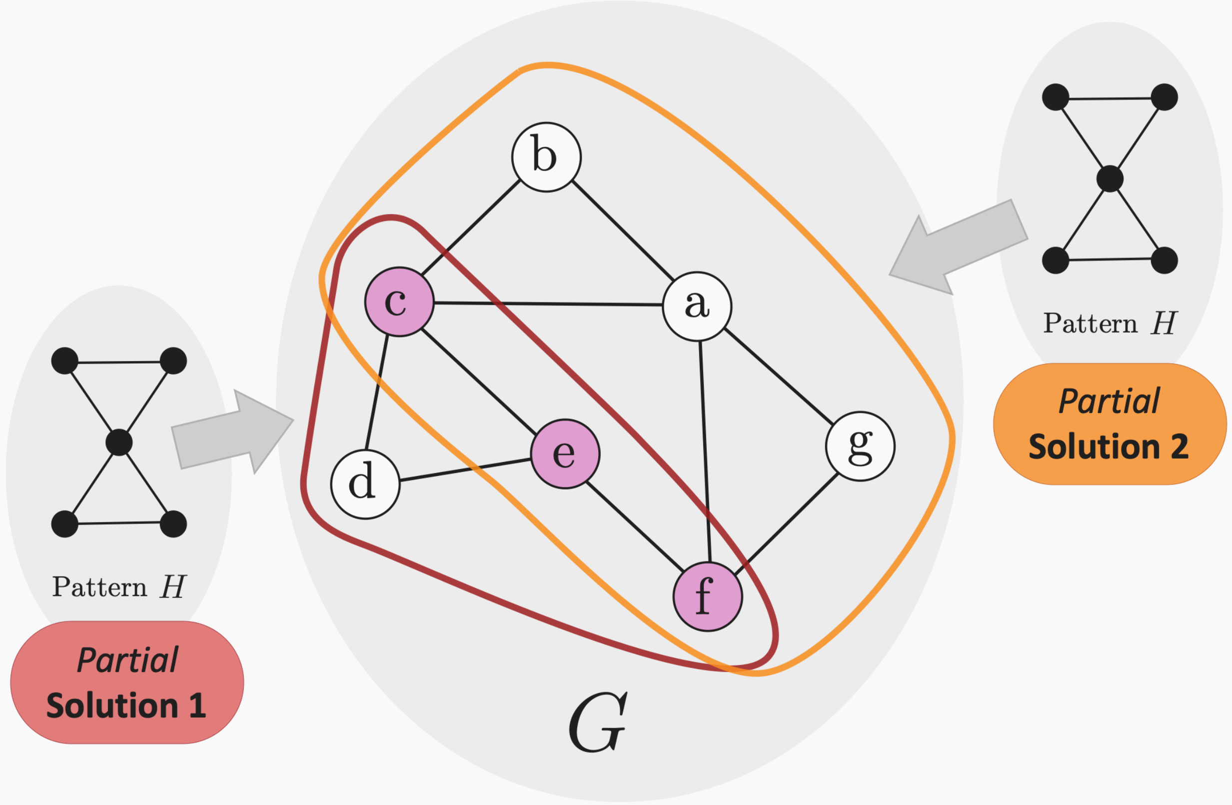
Partial **Solution 1**

Partial **Solution 2**

Exponential in "shared" part

General $\Omega(n)$

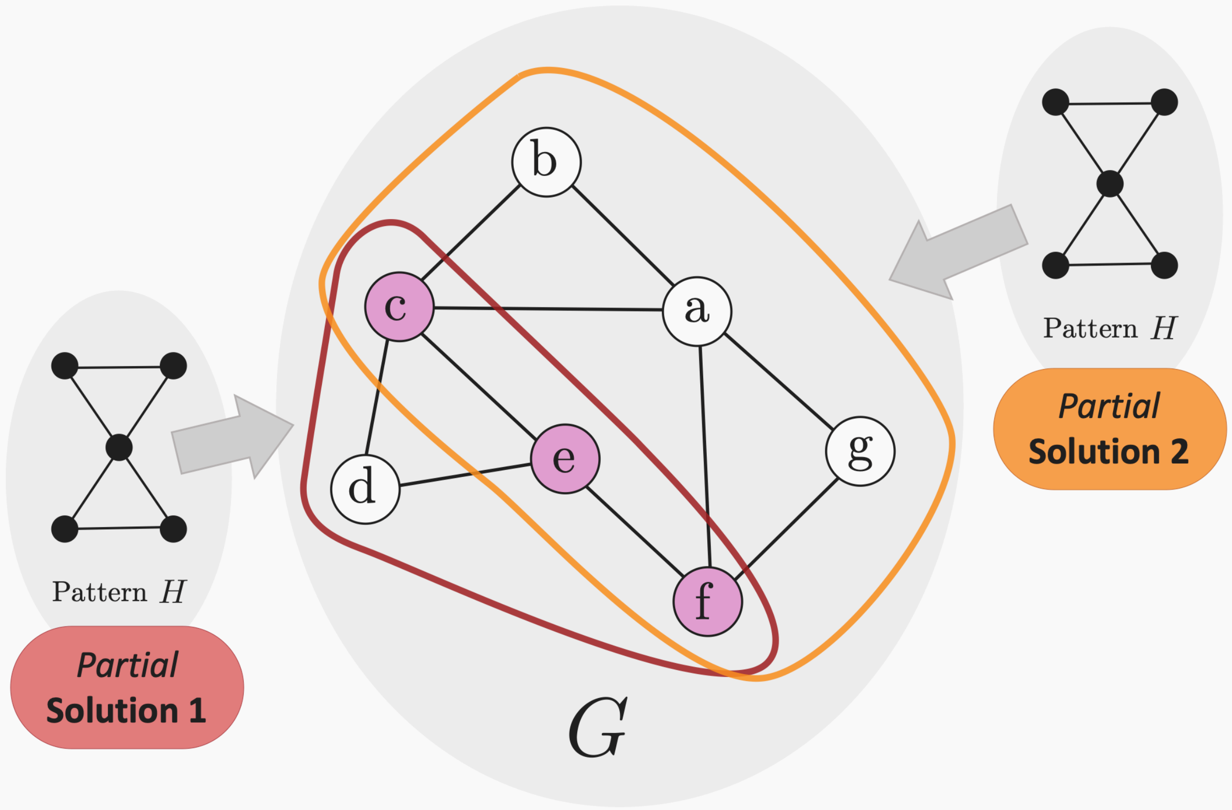
Dynamic Programming



Exponential in "shared" part

General	$\Omega(n)$
Planar	$\Theta(\sqrt{n})$

Dynamic Programming



Exponential in "shared" part

General	$\Omega(n)$
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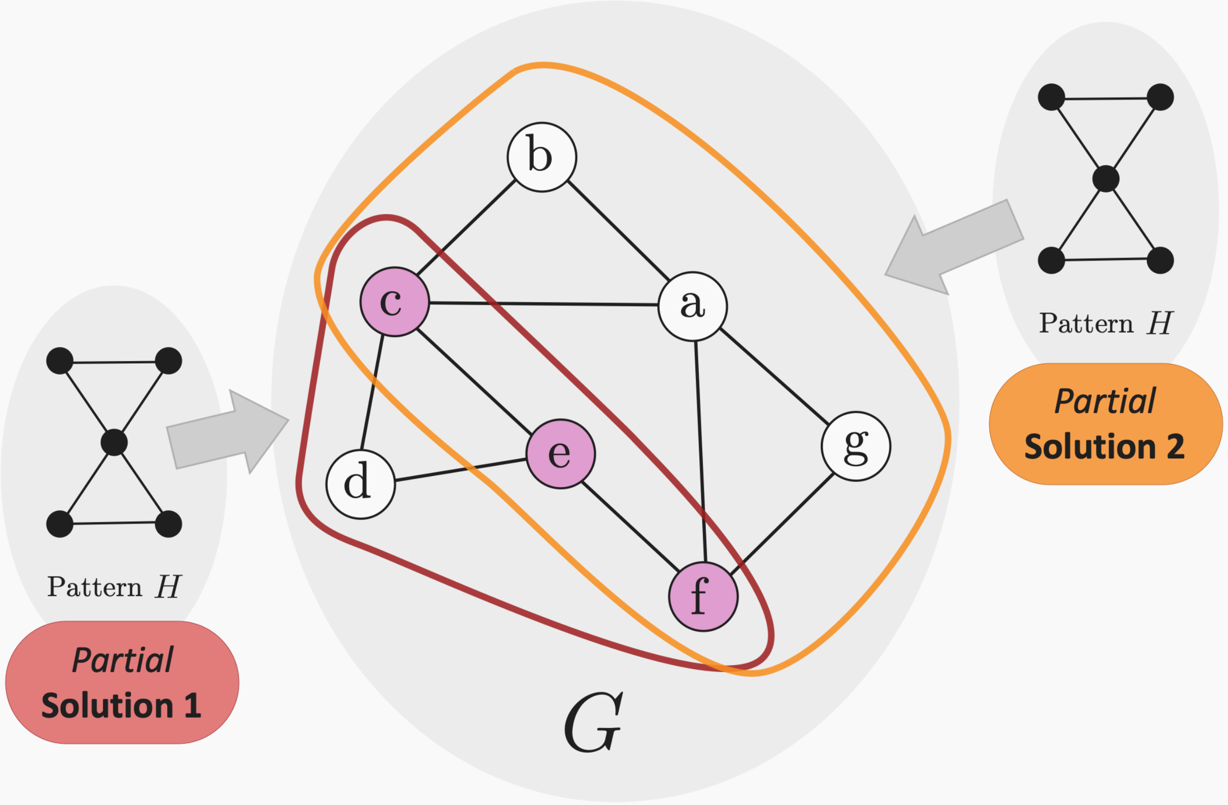
Planar, *diameter d*	$O(d)$

Dynamic Programming

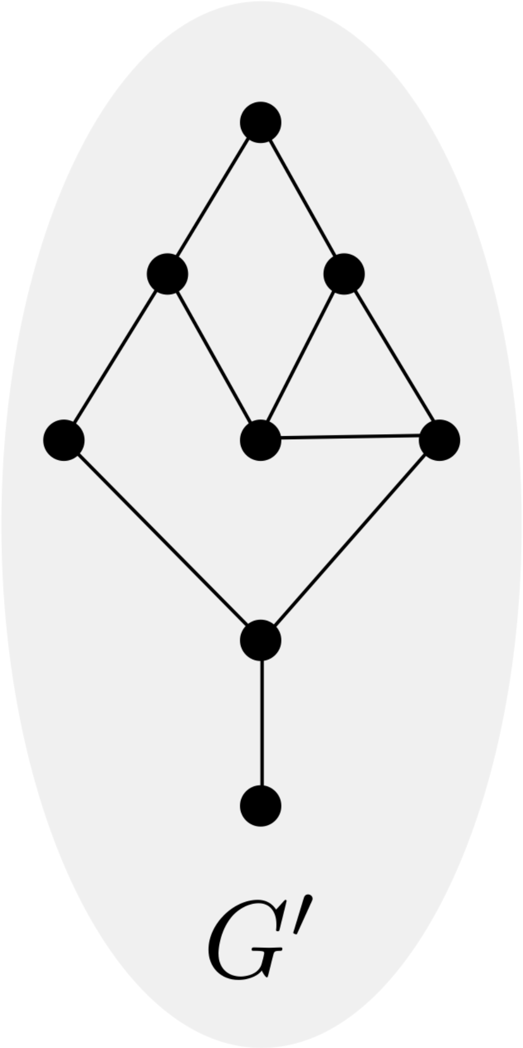


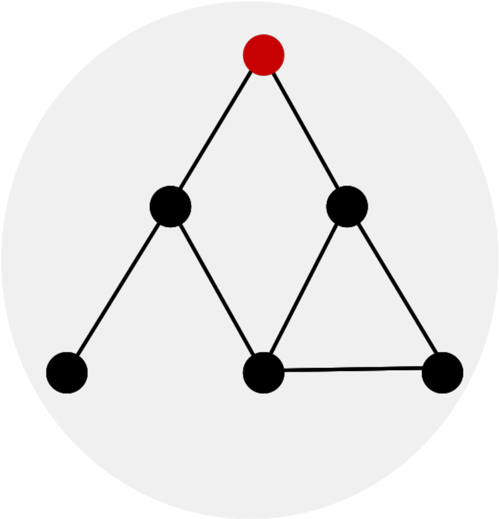
Exponential in "shared" part

General	$\Omega(n)$
Planar	$\Theta(\sqrt{n})$
Planar, *diameter d*	$O(d)$

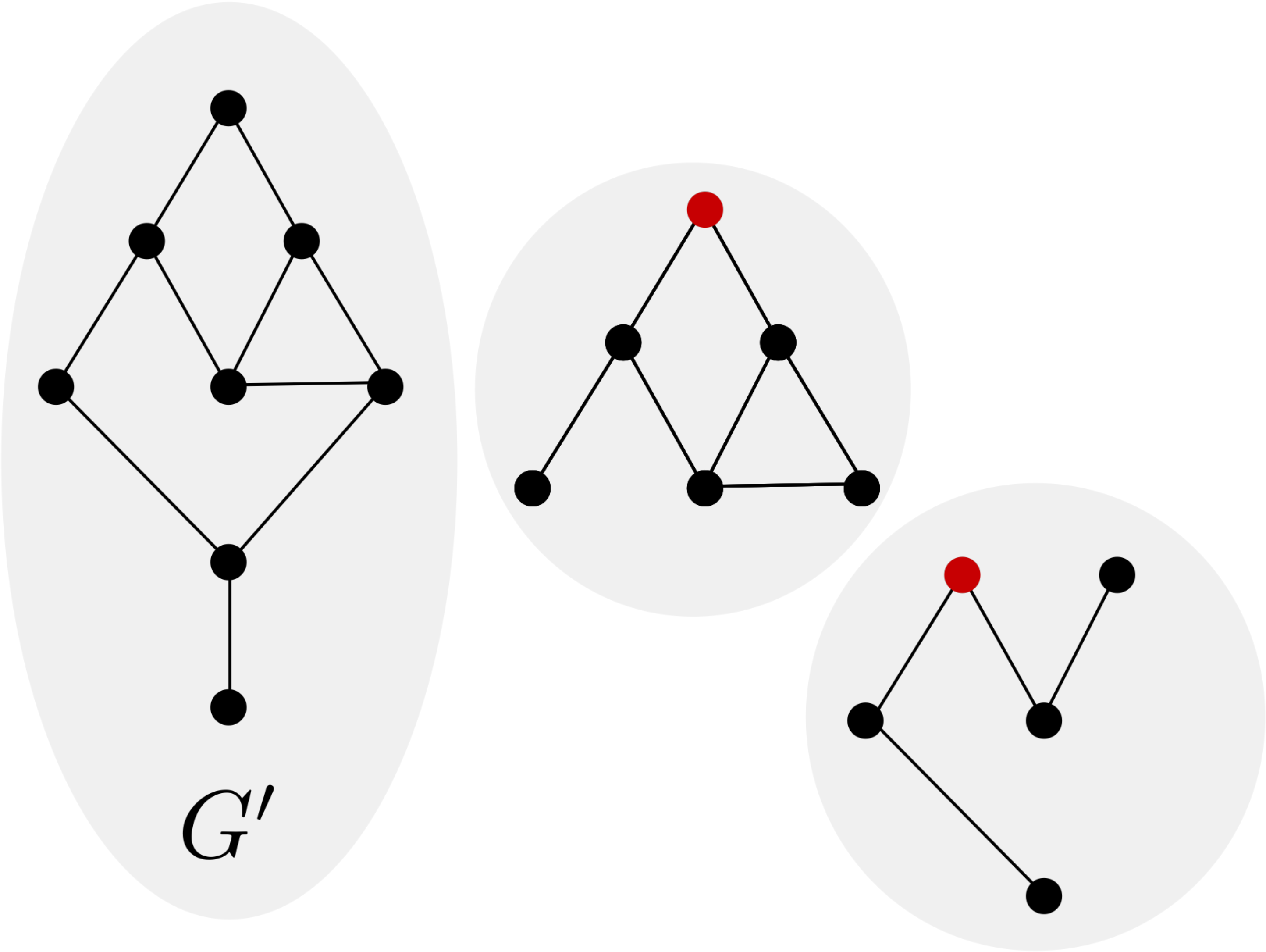
Check **diameter *k-1* subgraphs**

Naïve Covering





Naïve Covering



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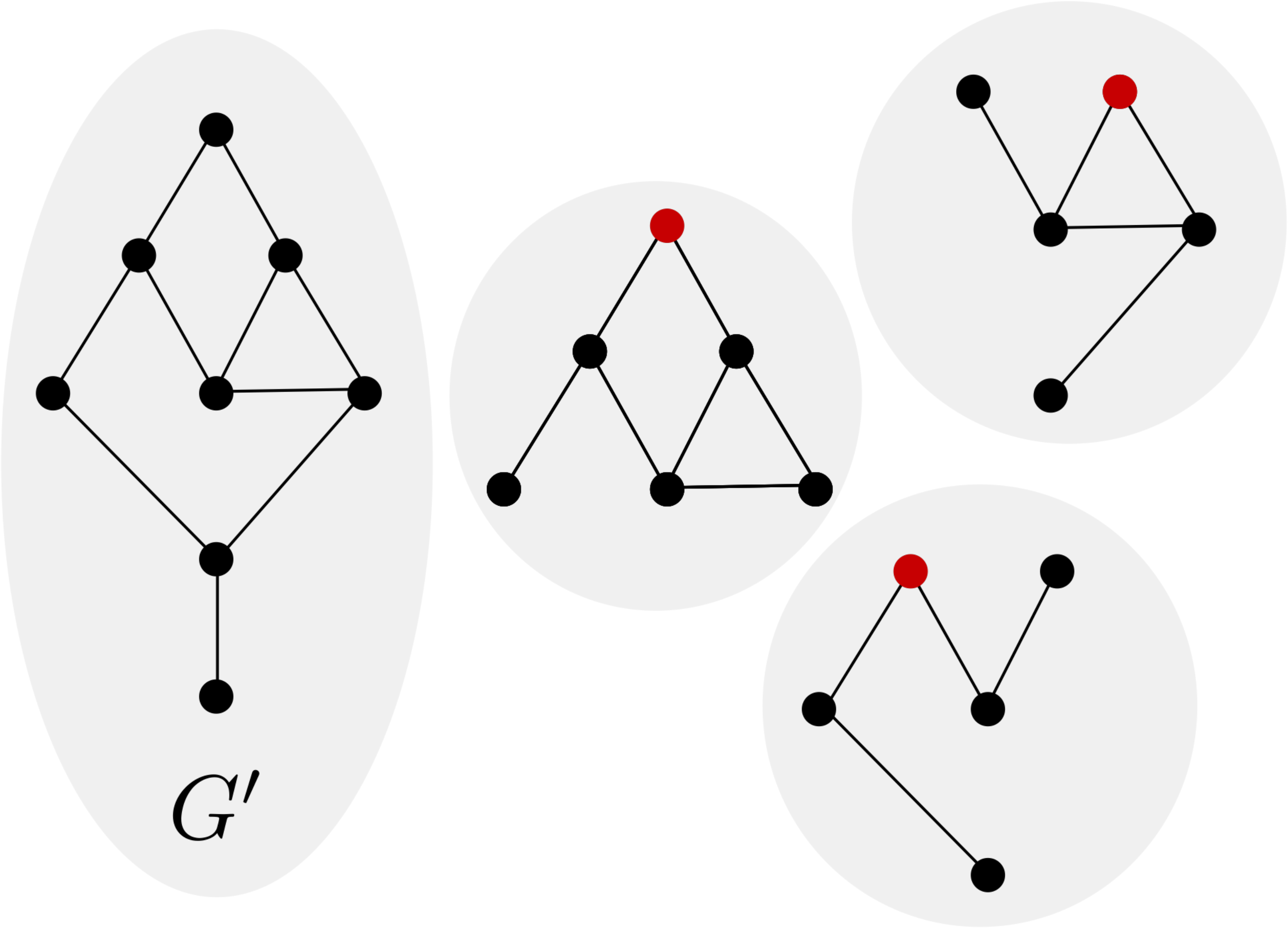

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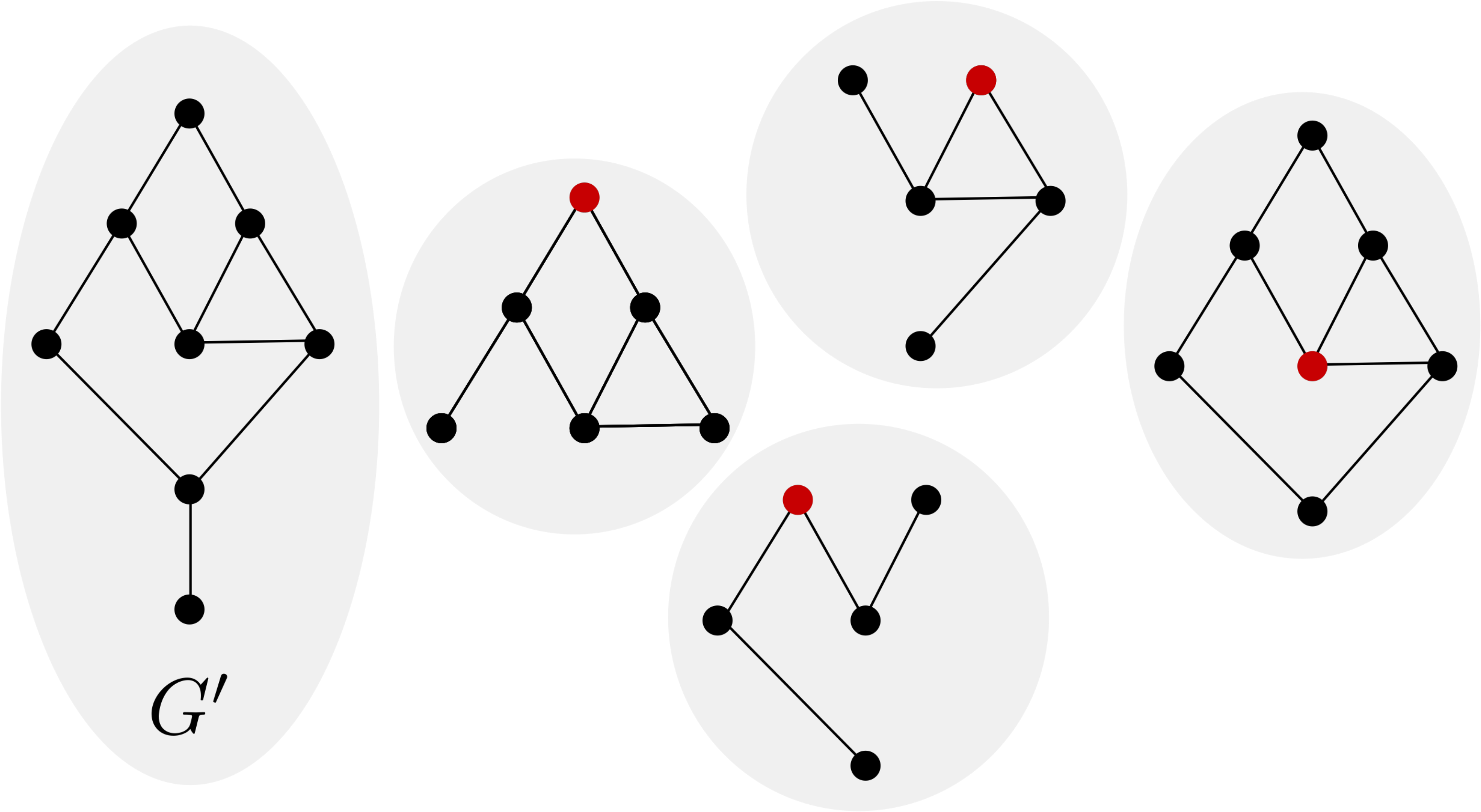

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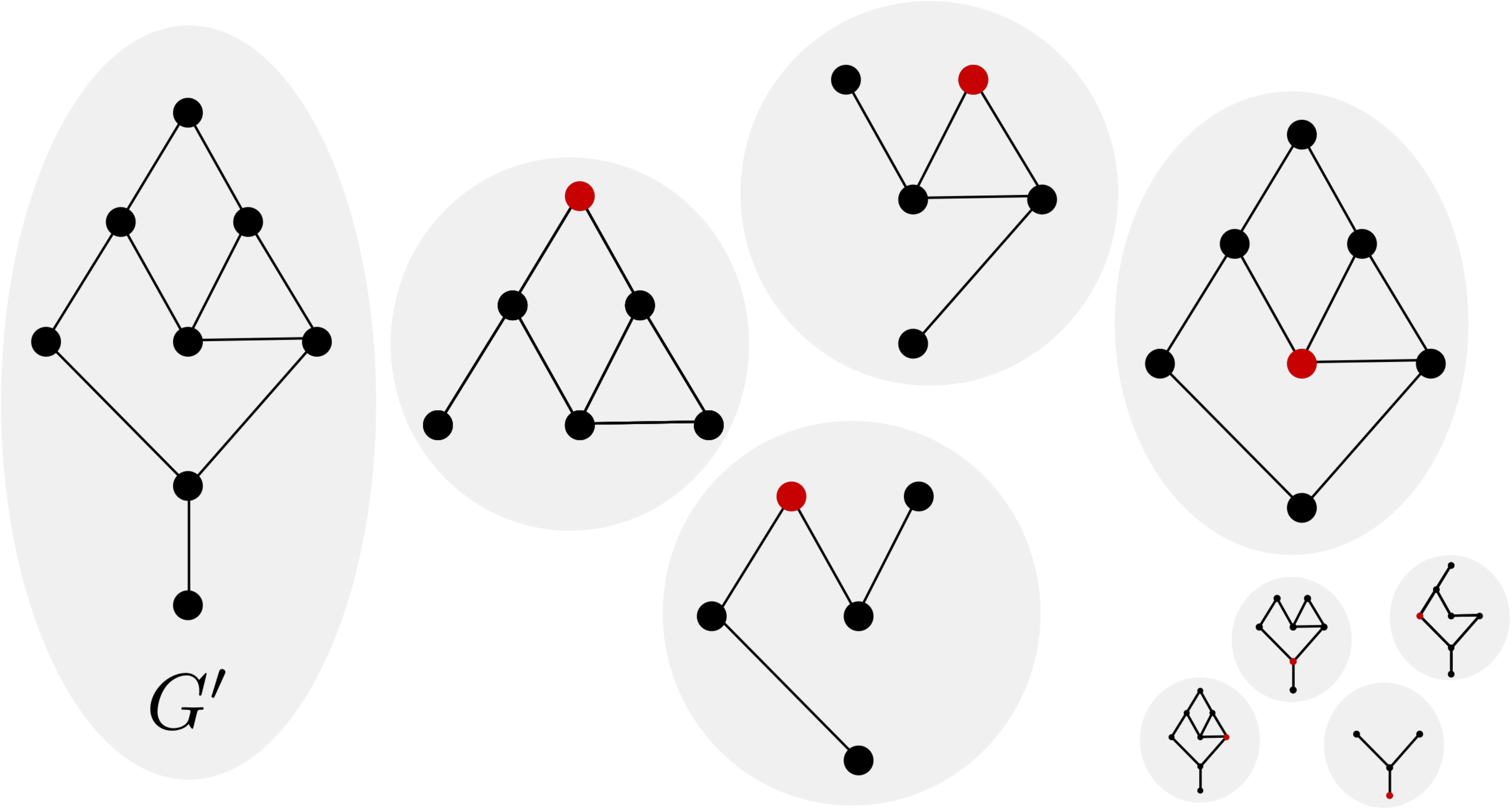

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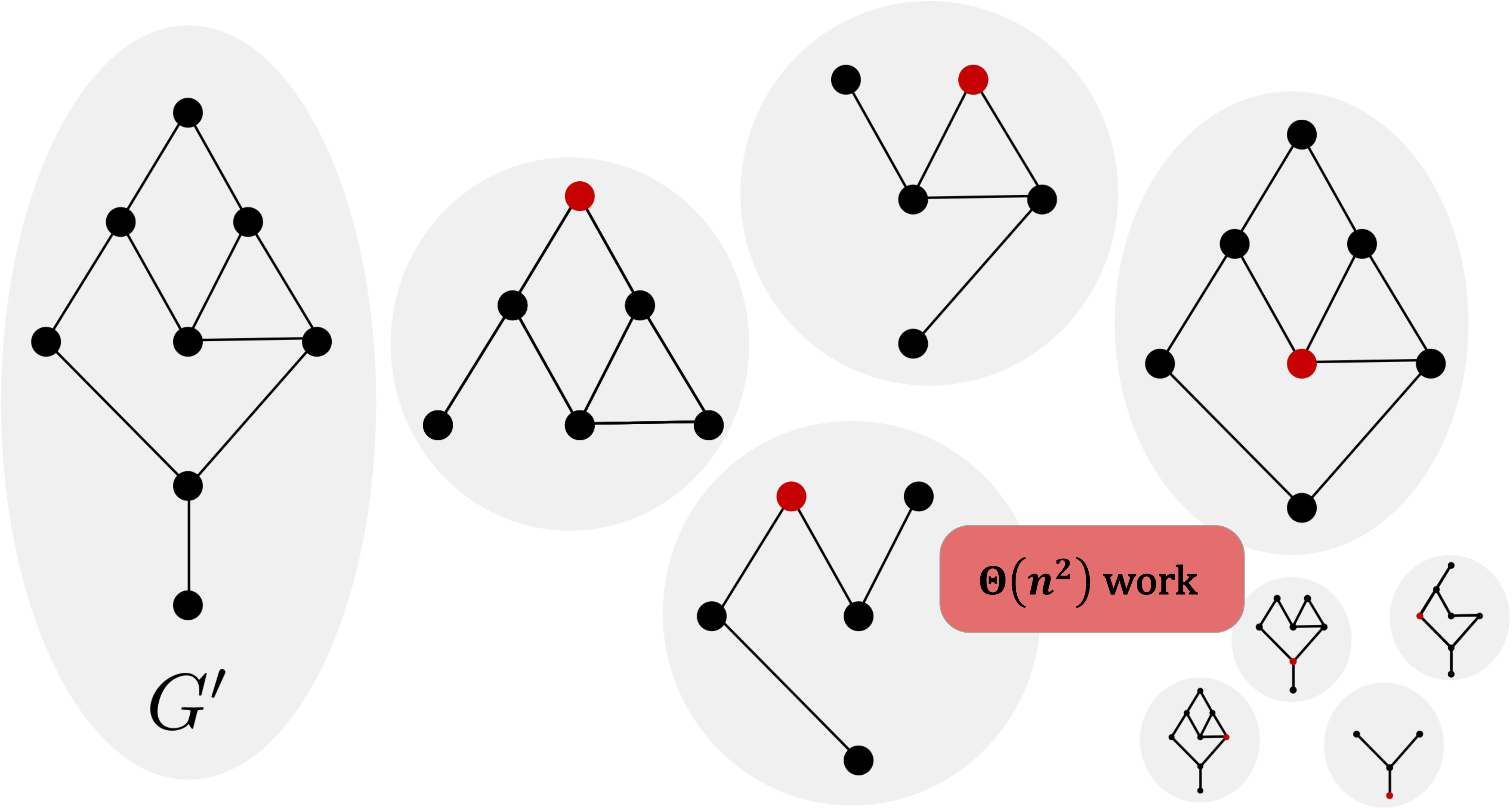
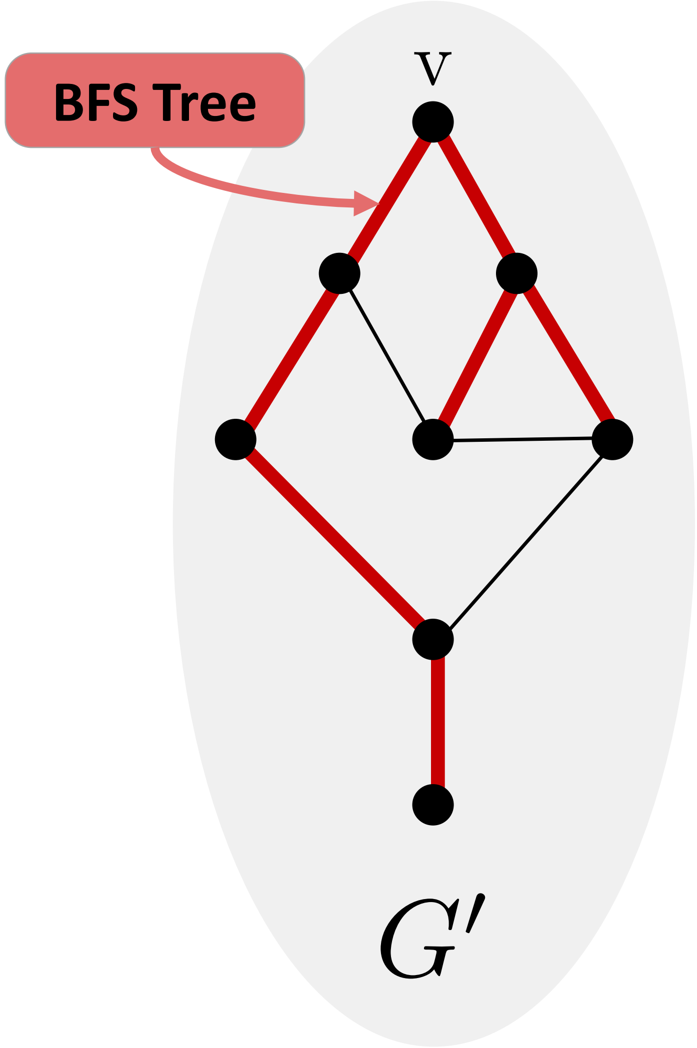


$\Theta(n^2)$ work

Work-Efficient Covering with BFS



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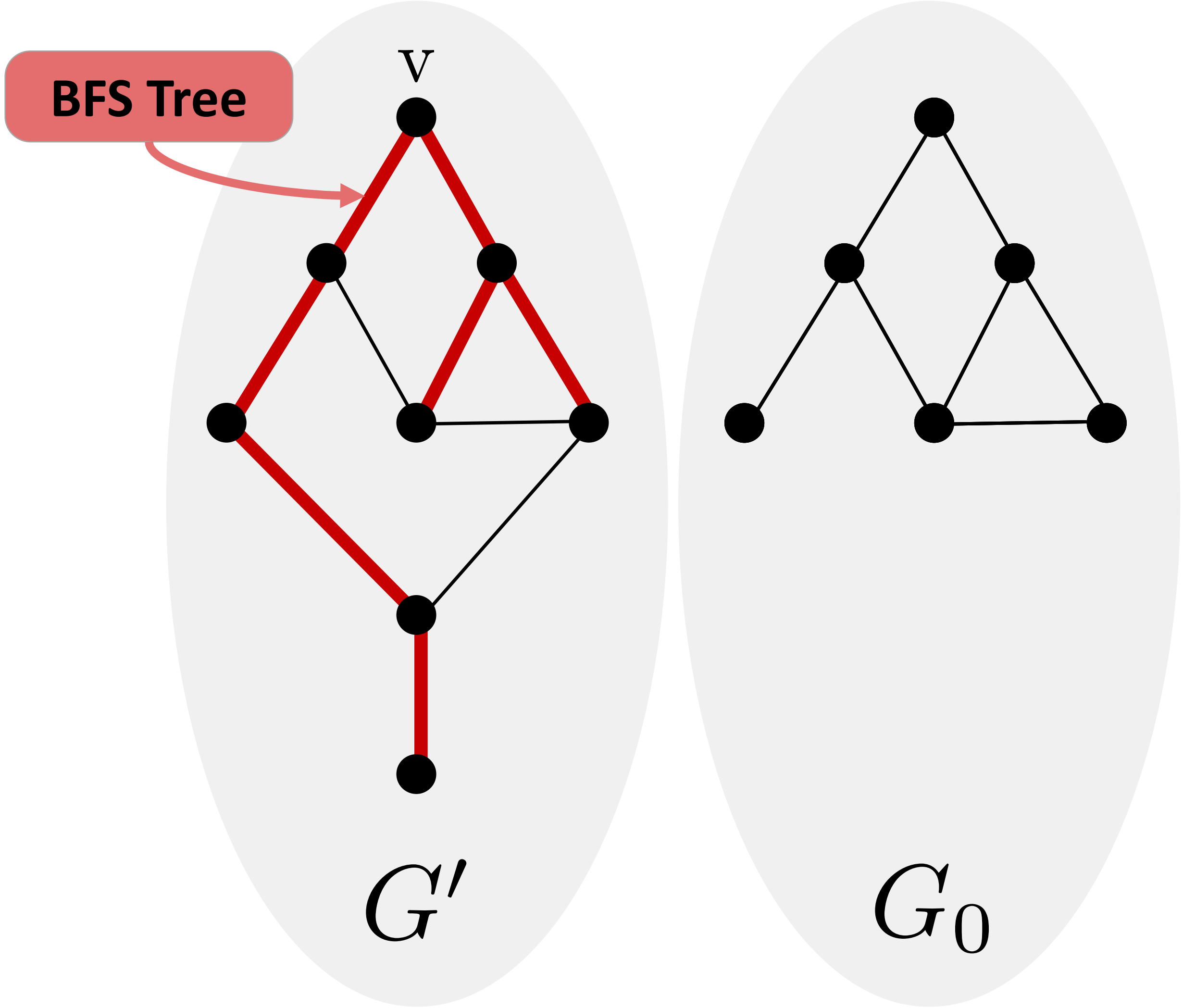

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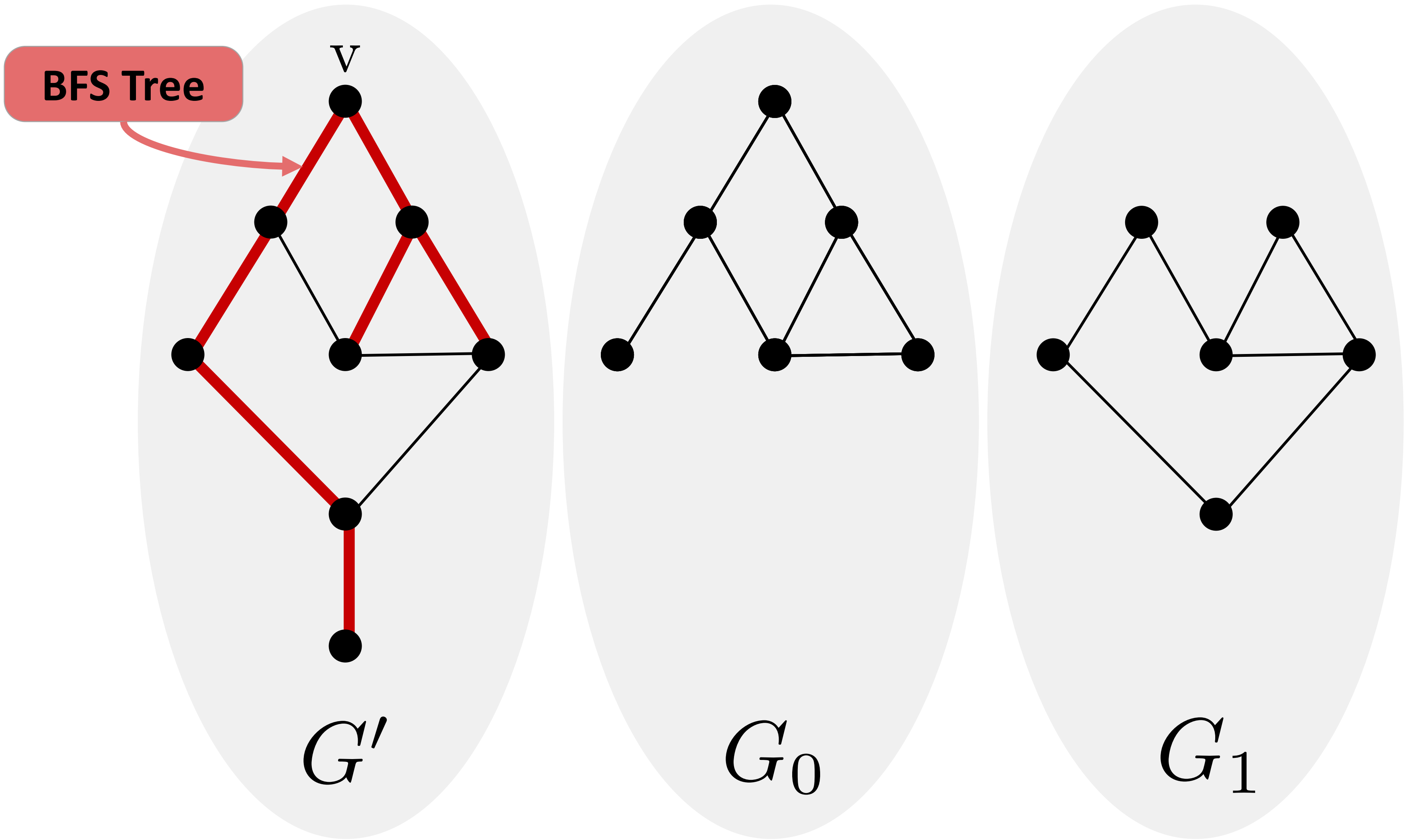

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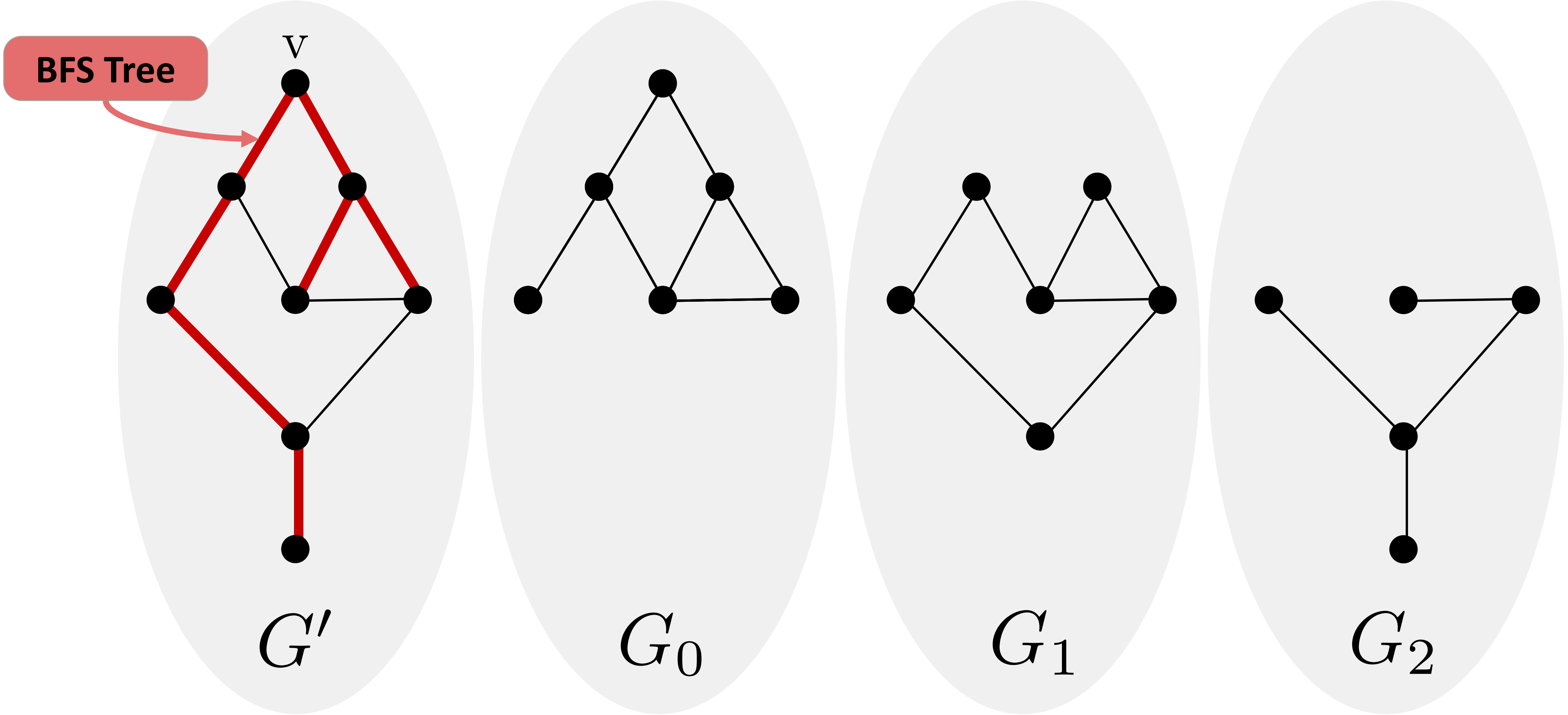
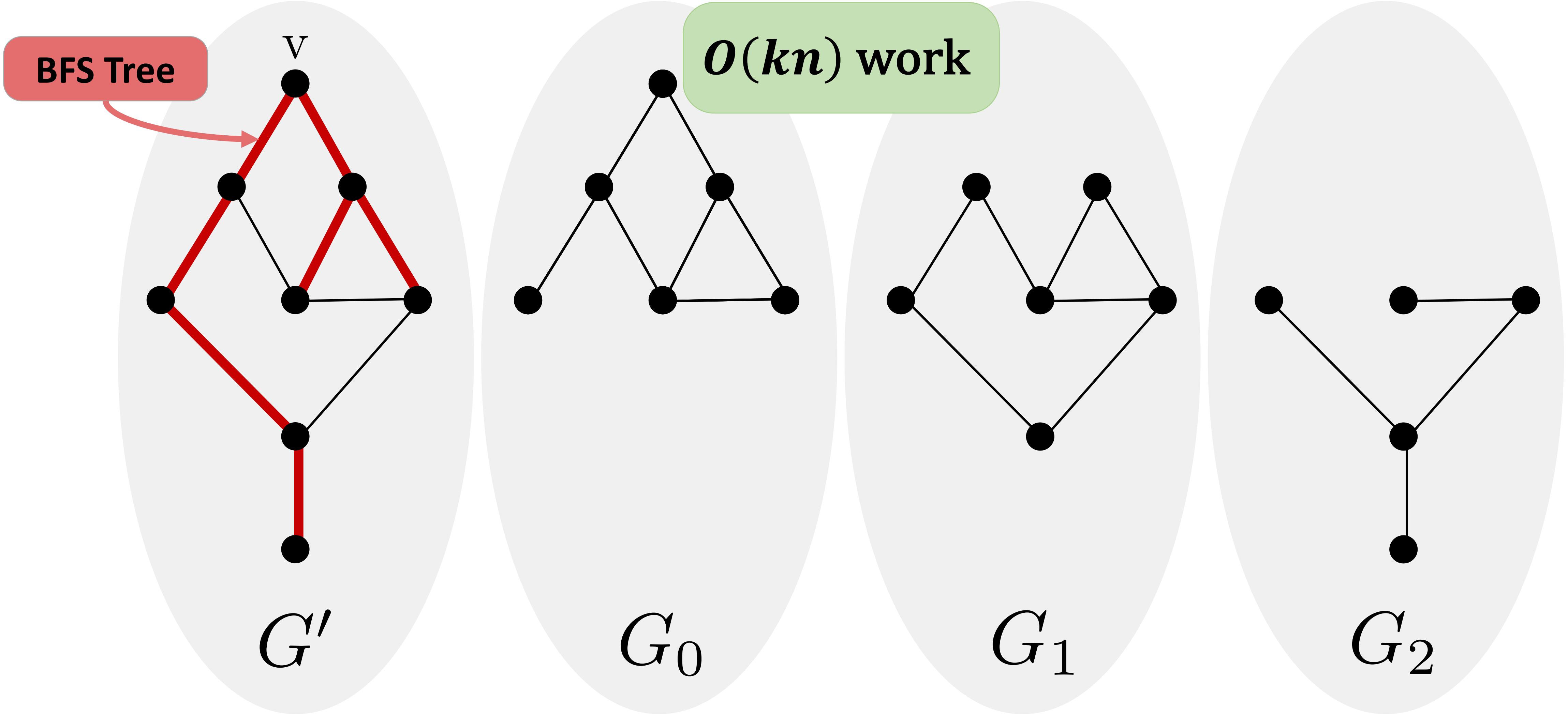


Work-Efficient Covering with BFS

$O(kn)$ work



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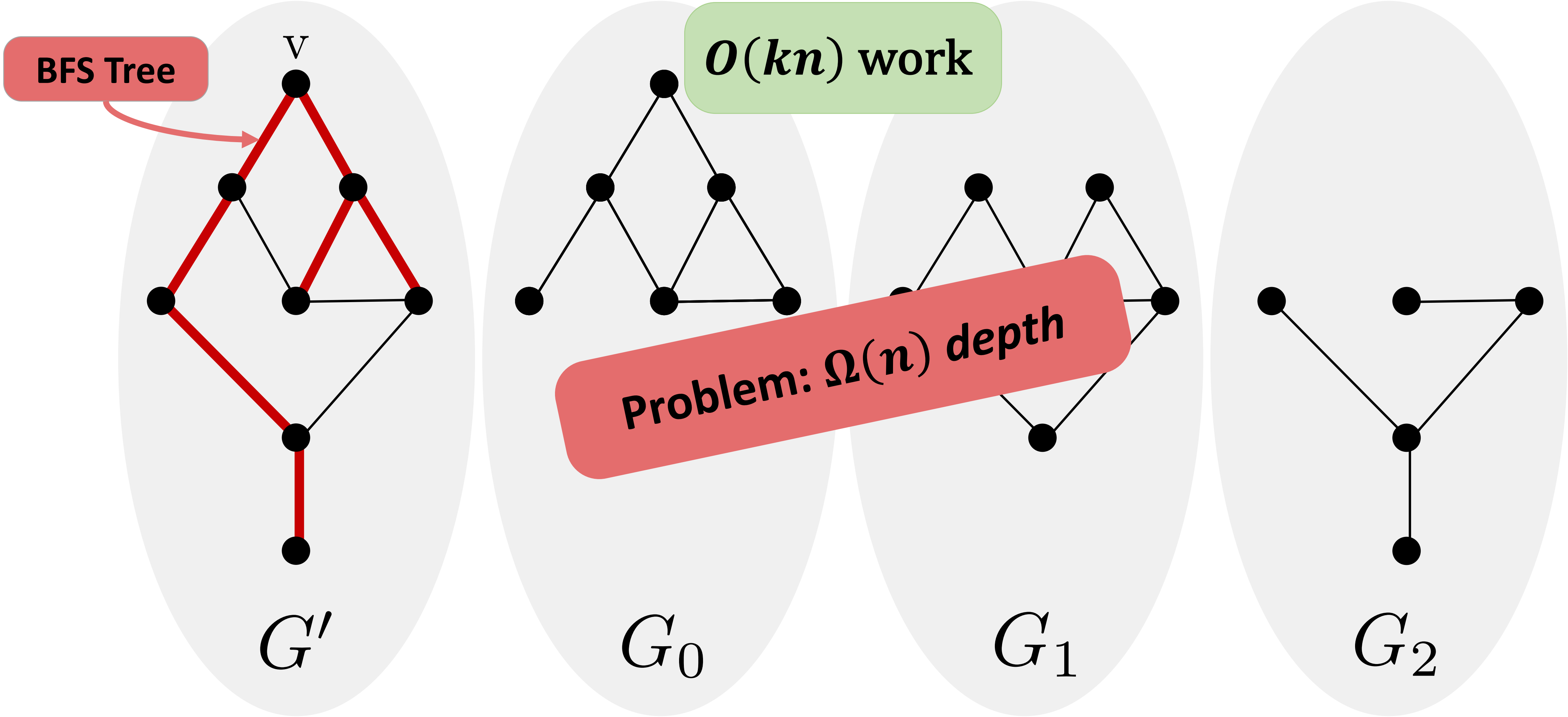
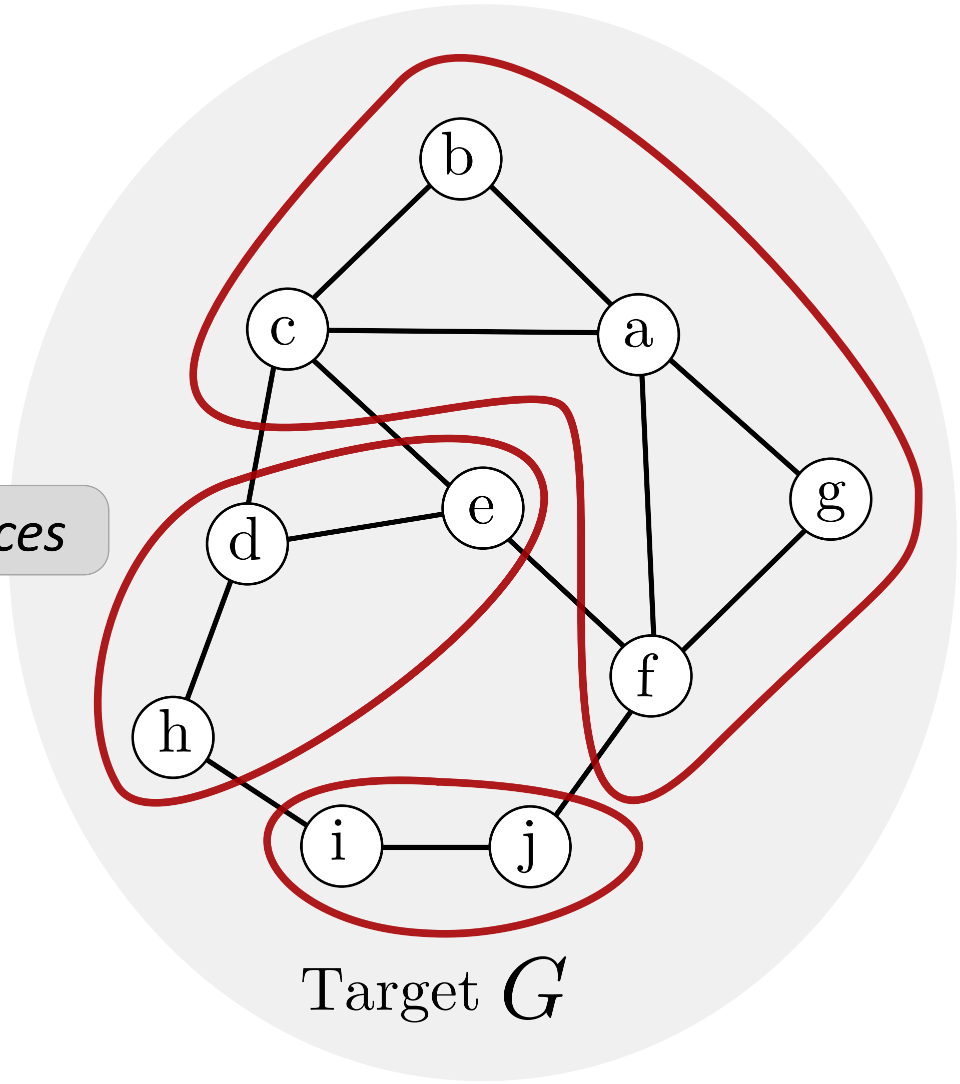


Low-Diameter Decomposition

Miller et al. 2015

***n* vertices**

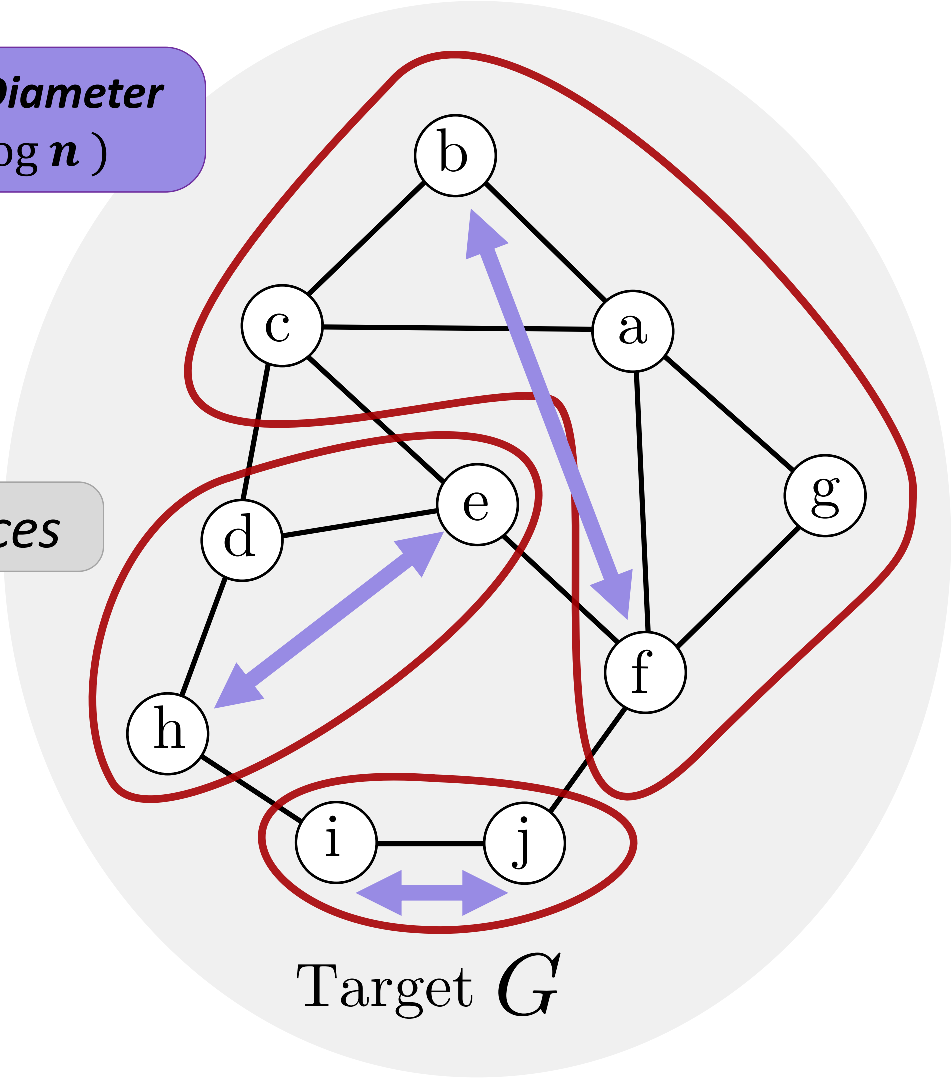


Low-Diameter Decomposition

Miller et al. 2015

Cluster Diameter
$O(\mathbf{k} \log n)$

***n* vertices**



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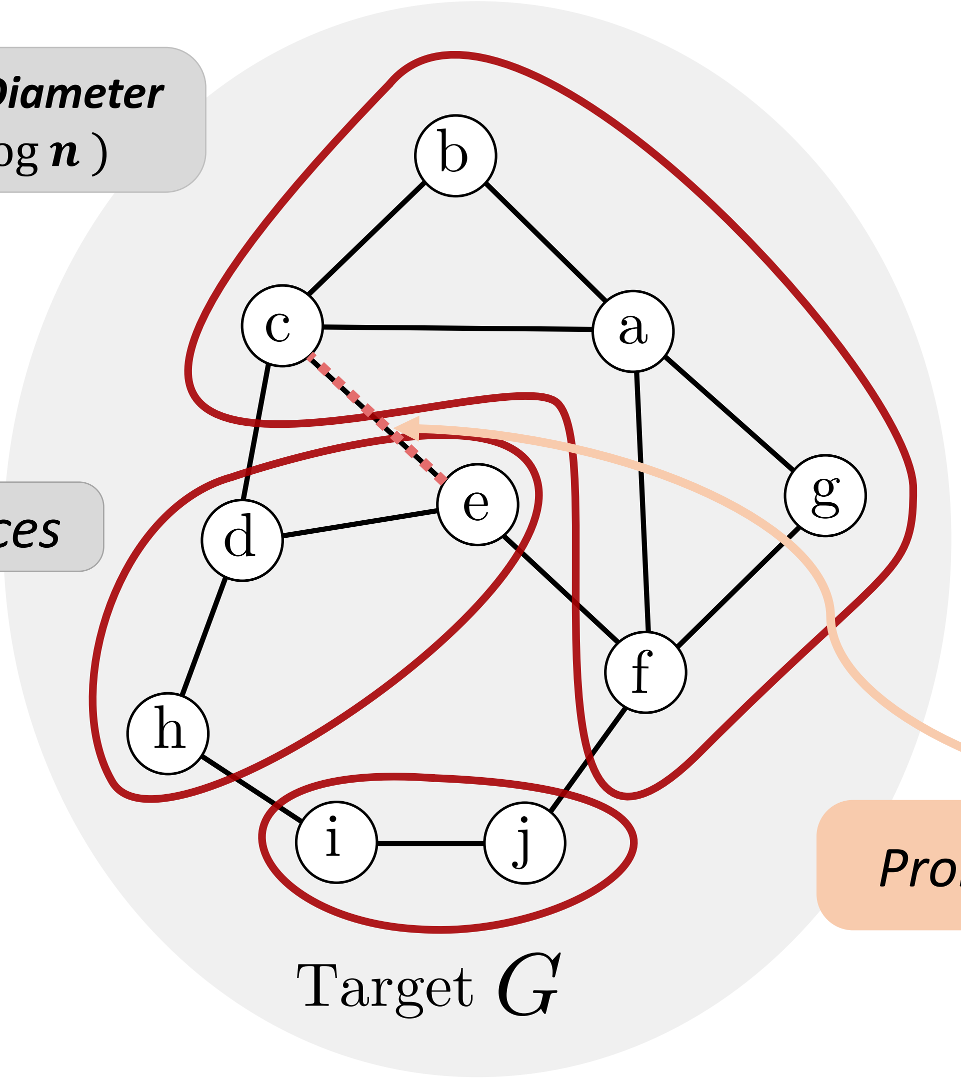

Probability a particular **edge crosses** $\leq \frac{1}{2k}$

Low-Diameter Decomposition

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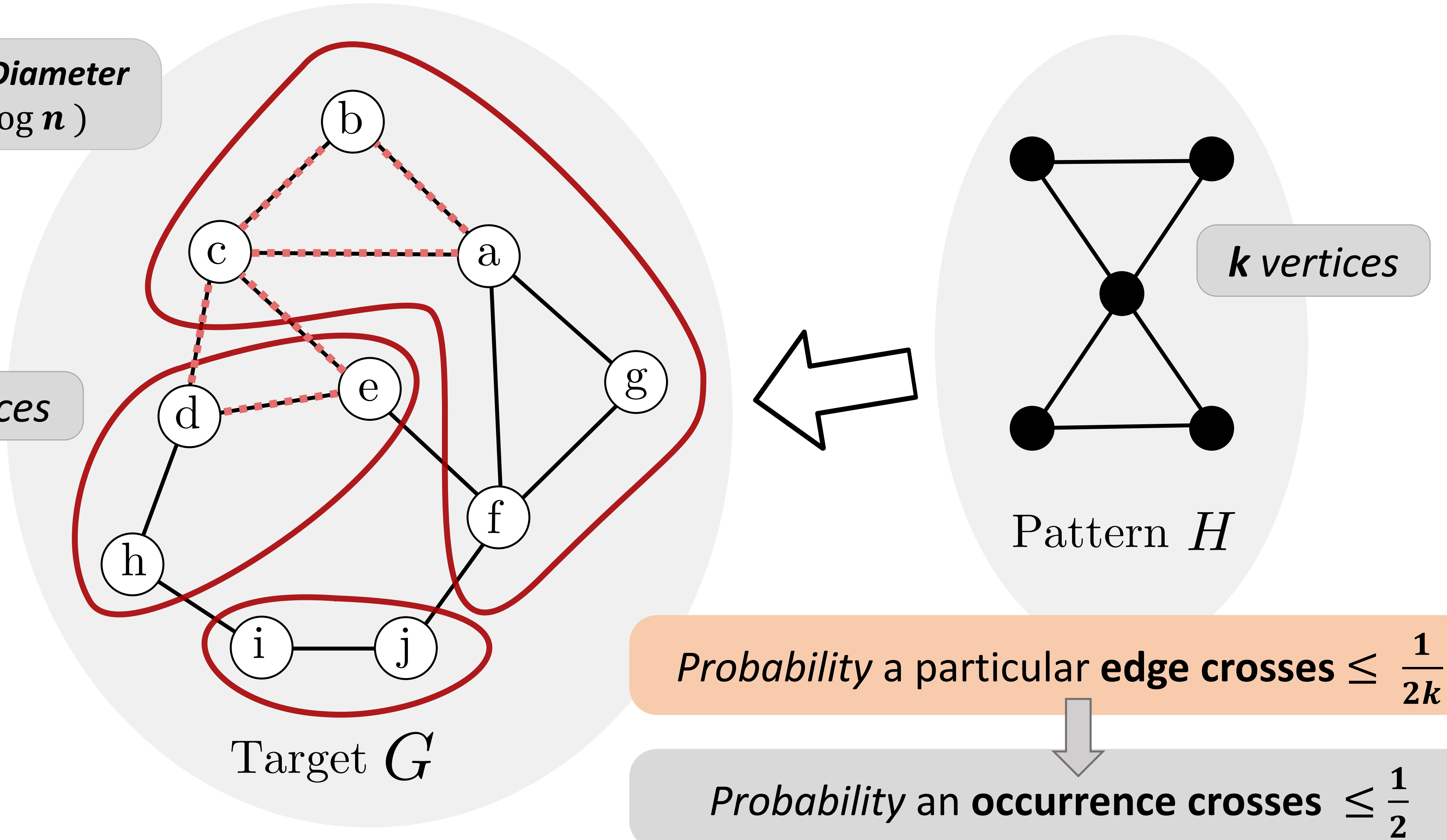

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Probability an **occurrence crosses** $\leq \frac{1}{2}$

Low-Diameter Decomposition

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***n* vertices**

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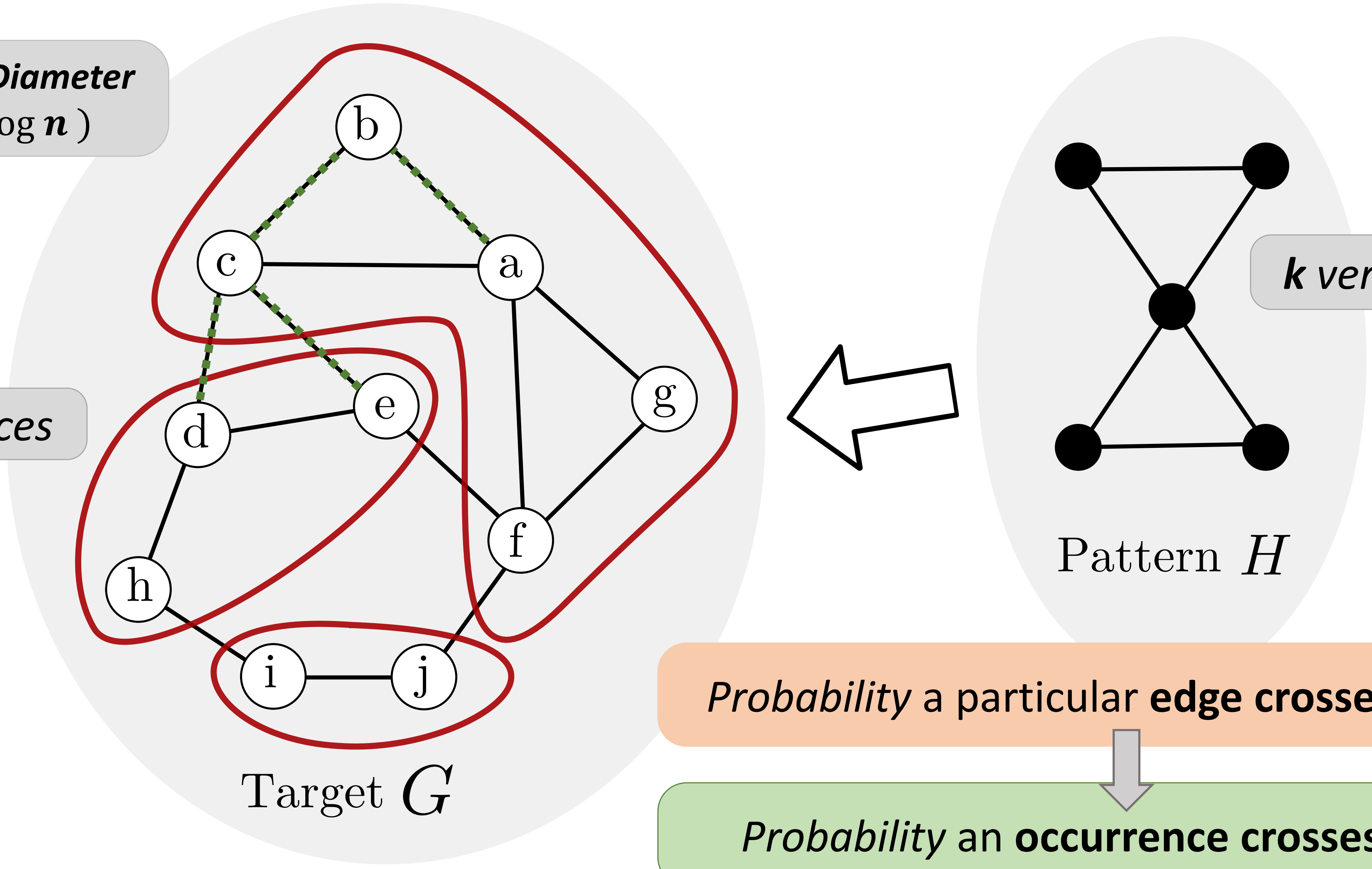

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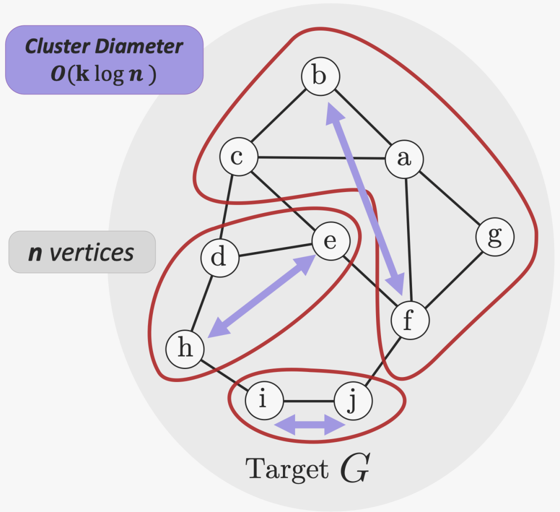
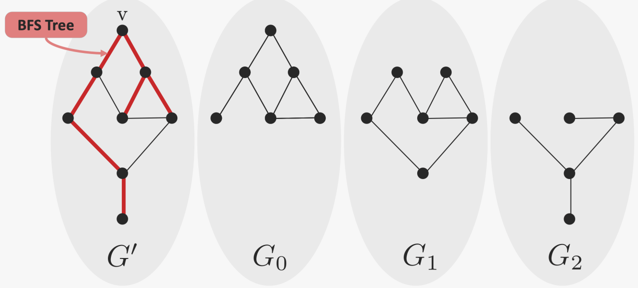
Planar Subgraph Isomorphism

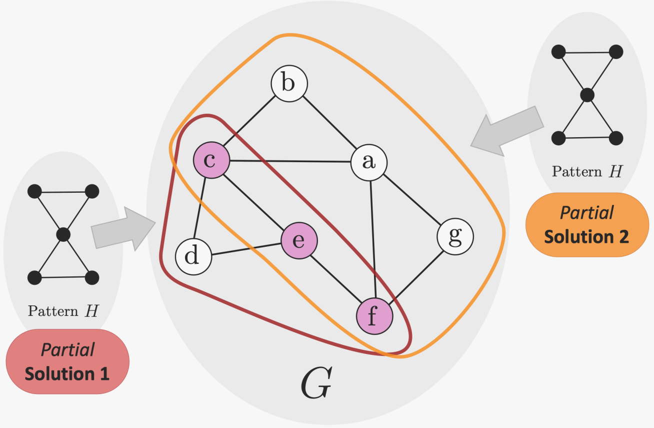
Low Diameter Decomposition



Covering with BFS



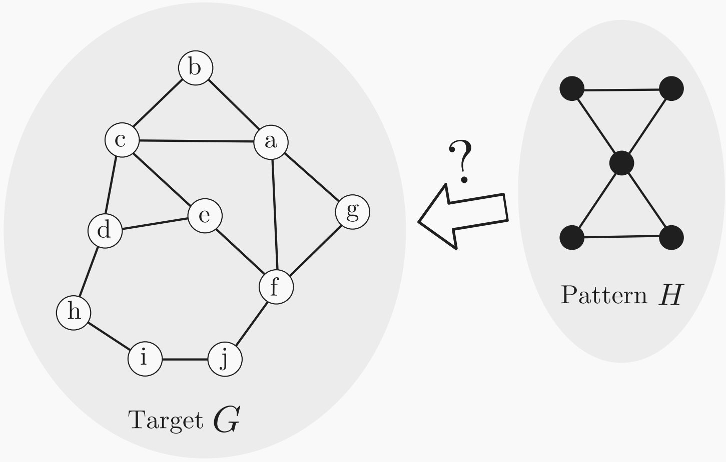
Dynamic Programming



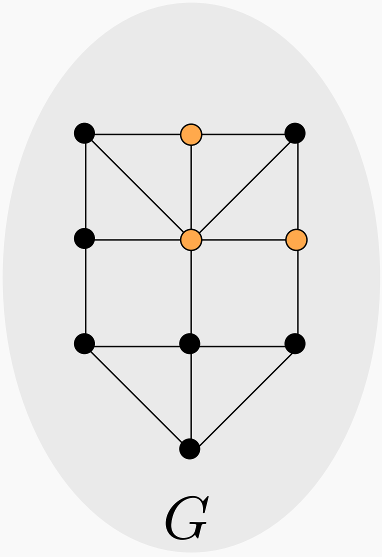
Low Diameter Decomposition	Covering with BFS	Dynamic Programming
$O(n)$ work	$O(kn)$ work	$O(k^{3k+1}n)$ work
$O(k \log n)$ depth	$O(k \log n)$ depth	$O(k \log n)$ depth

$O(\log n)$ repetitions

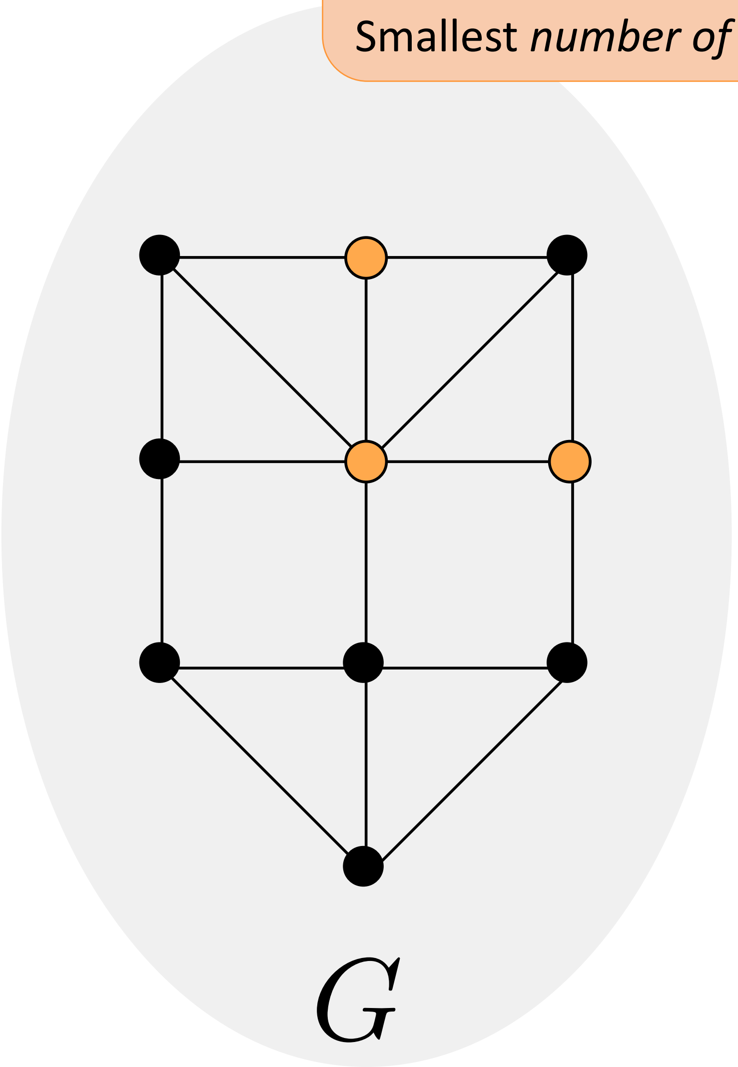
Subgraph Isomorphism



Minimum Vertex Cut



Minimum Vertex Cut
Smallest *number of vertices* whose ***removal disconnects the graph***



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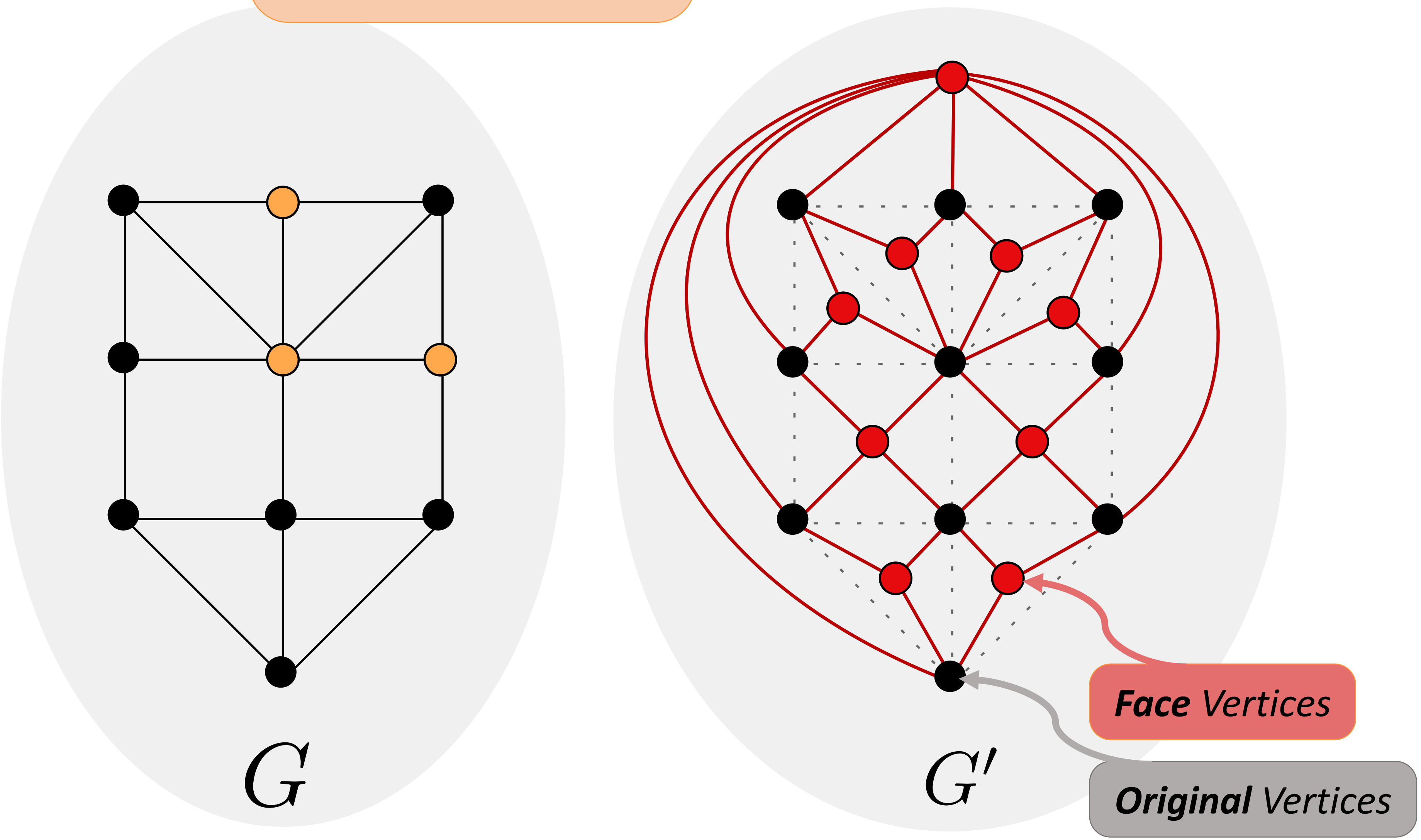
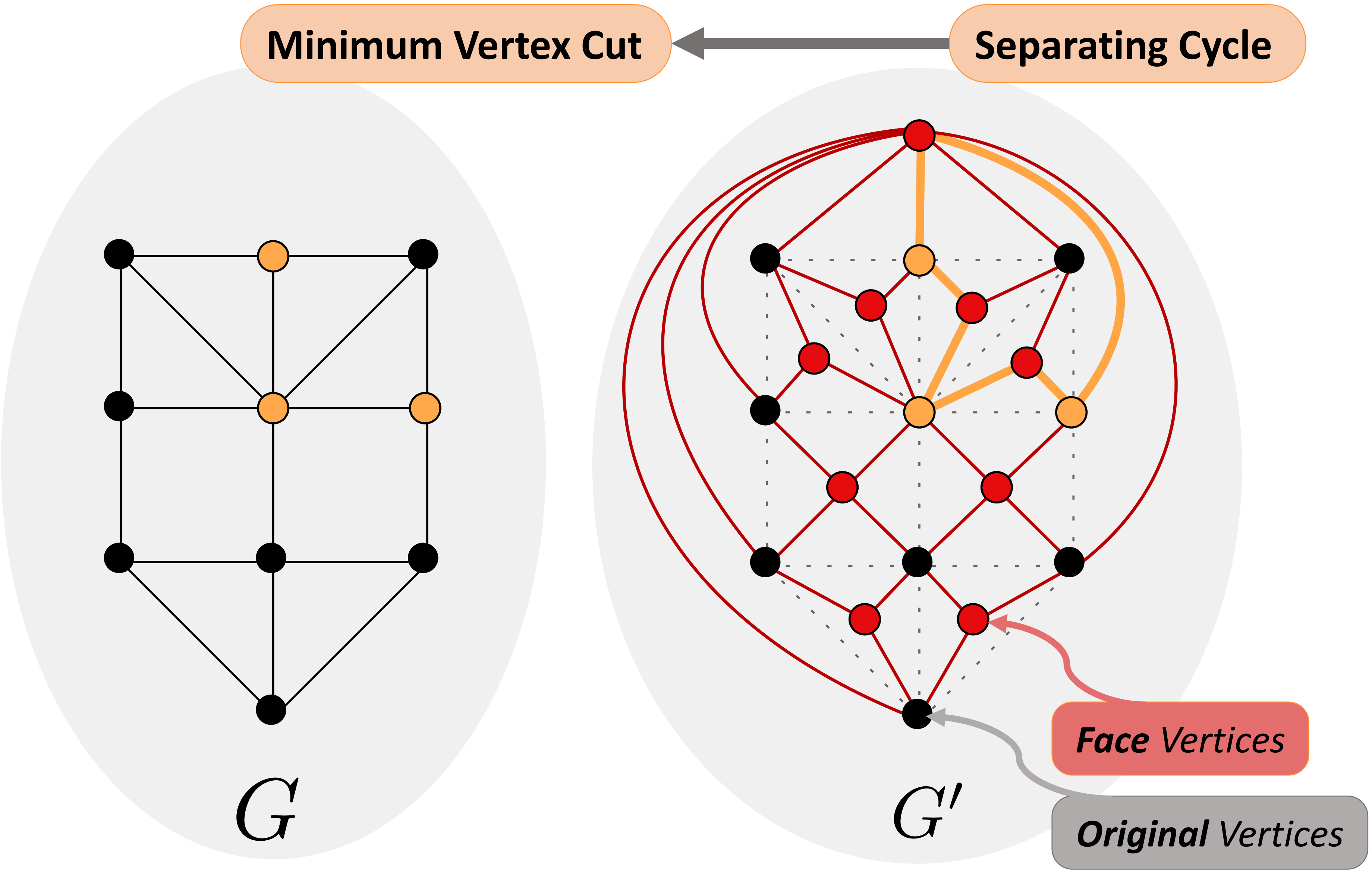
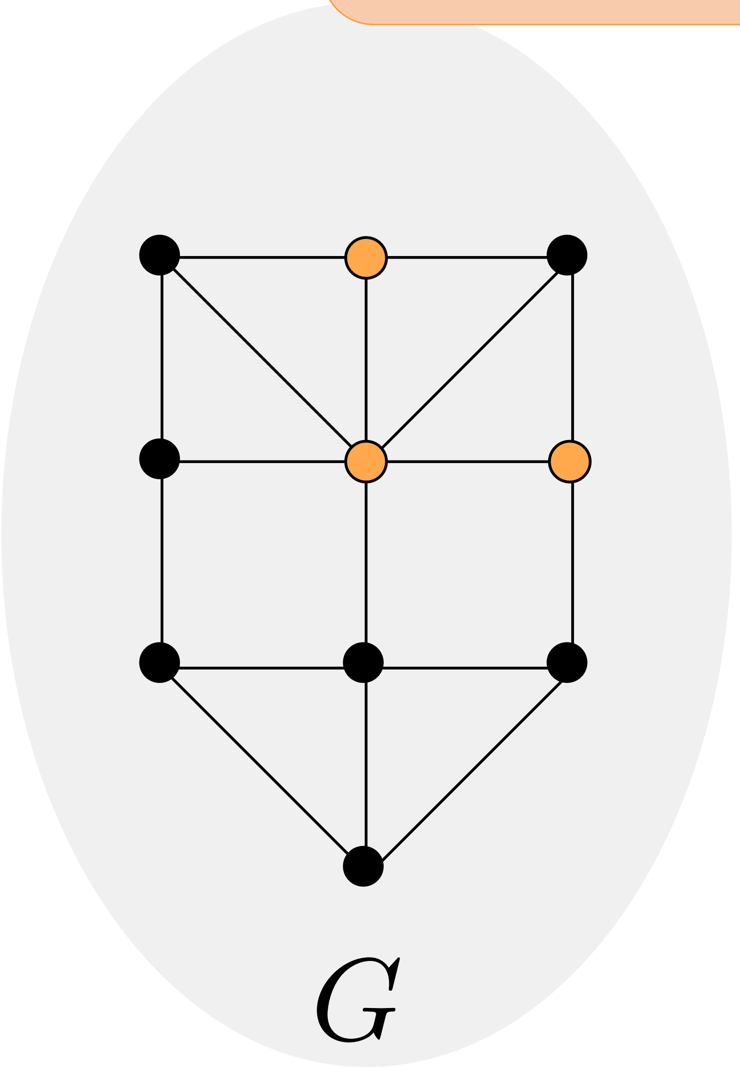
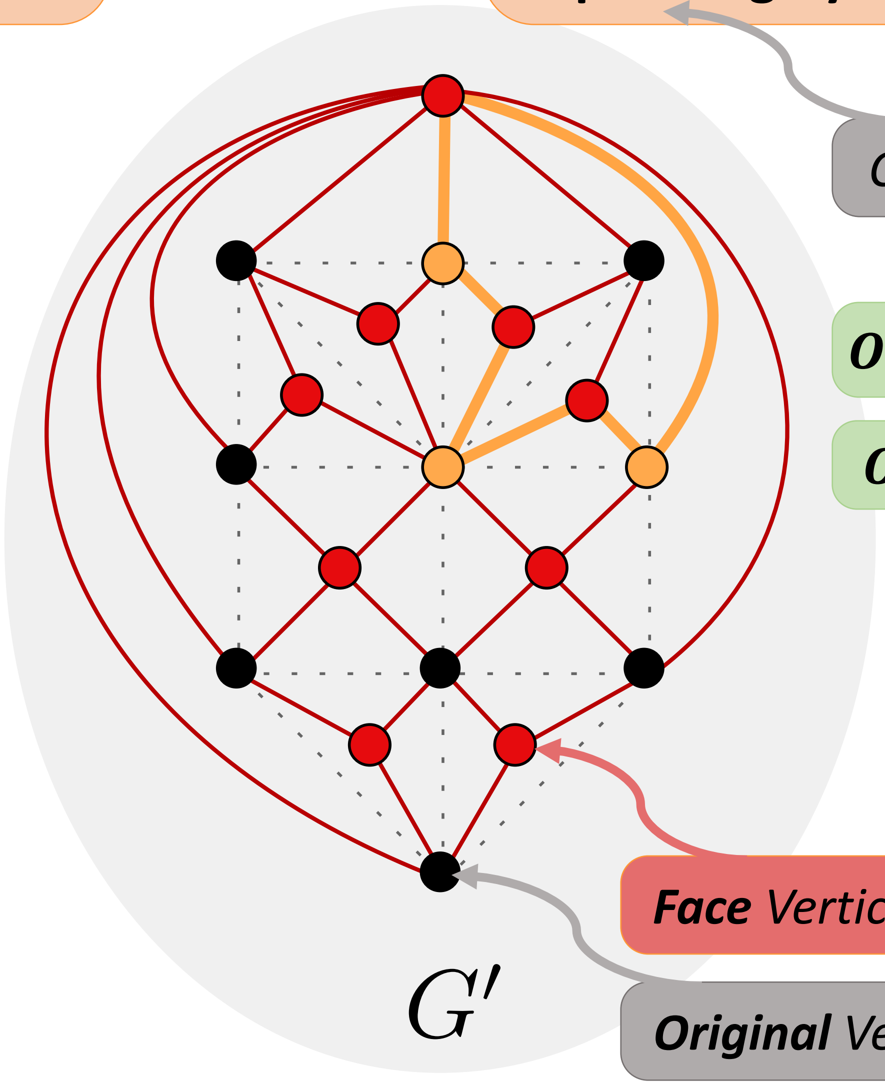



Minimum Vertex Cut
Separating Cycle
G
G'
Face Vertices
Original Vertices

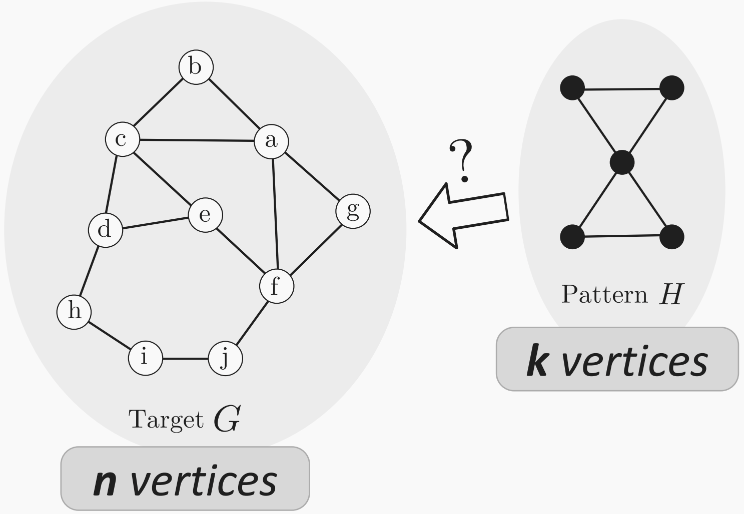
Minimum Vertex Cut ← Separating Cycle





Conclusion

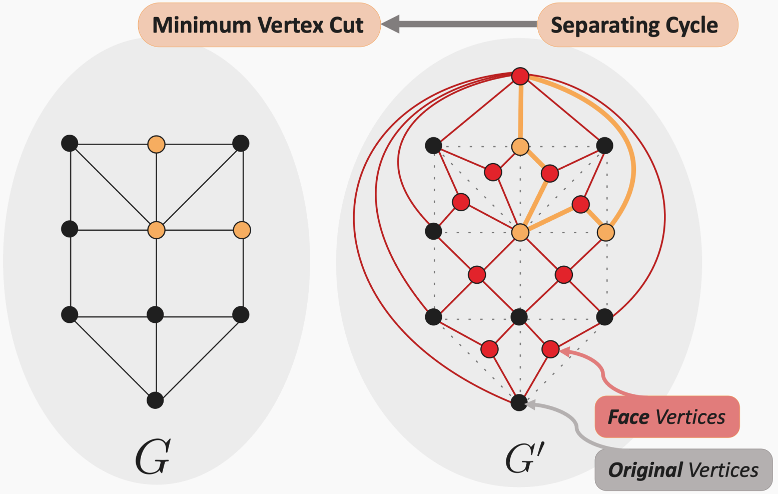
Subgraph Isomorphism



$O(k^{3k+1} n \log n)$ work

$O(k \log n)$ depth

Minimum Vertex Cut



$O(n \log n)$ work

$O(\log n)$ depth

Conclusion

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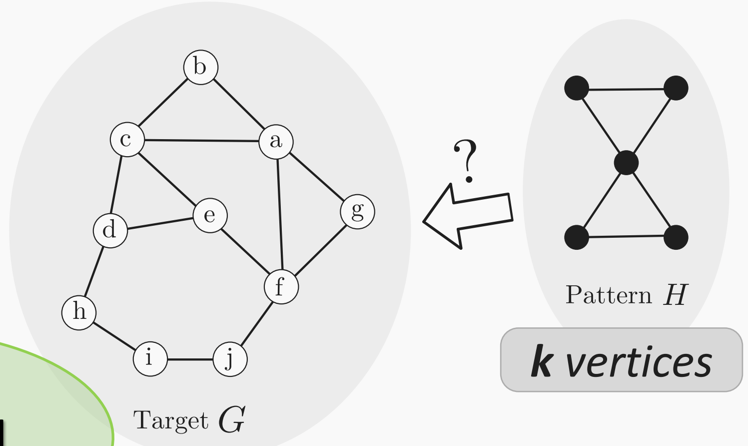


***n* vertices**

***k* vertices**

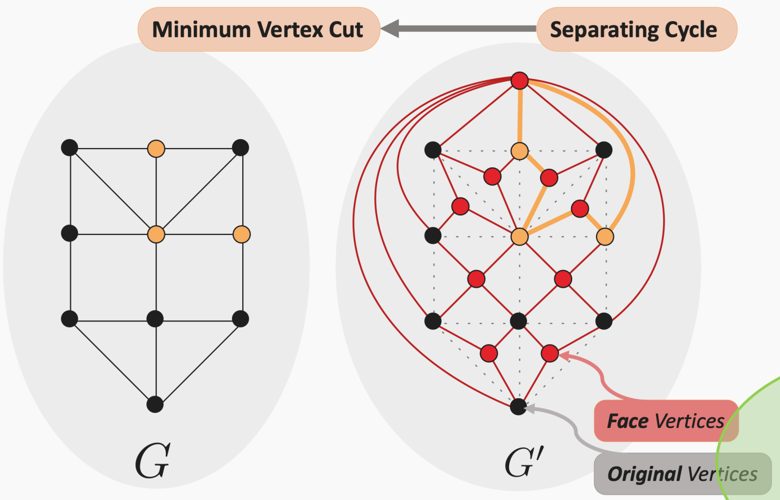
Singly exponential in k?

$O(k^{3k+1} n \log n)$ work

$O(k \log n)$ depth

Polylog in k?

Minimum Vertex Cut



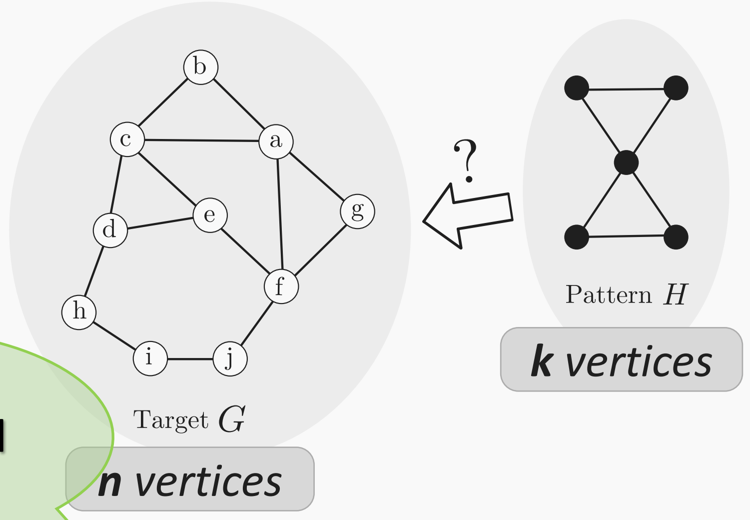
Linear in n?

$O(n \log n)$ work

$O(\log n)$ depth

Conclusion

Subgraph Isomorphism



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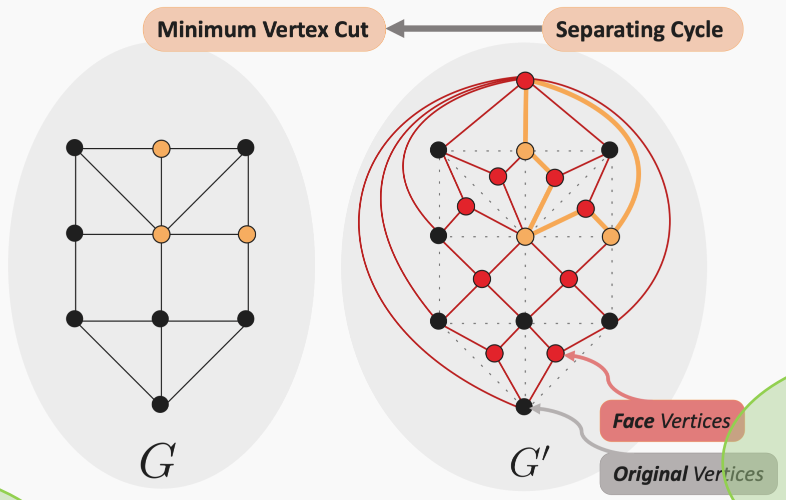
Polylog in k?

Other Implications?

Minimum Vertex Cut



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